Exam 4 Solutions

Unless otherwise stated, all curves are oriented counterclockwise when viewed from above, and surfaces are oriented upward/outward.

1. Evaluate the following integrals:

   (a) $\int_C x^2\,dx + xy\,dy + z^2\,dz, \quad \boldsymbol{r}(t) = \langle \sin t, \cos t, t^2\rangle, \quad 0 \le t \le \pi.$

$$\int_0^\pi \sin^2 t\cos t + \sin t\cos t(-\sin t) + t^4(2t)\,dt = \int_0^\pi 2t^5\,dt = \left[\frac{1}{3}t^6\right]_0^\pi = \frac{\pi^6}{3}$$

   (b) $\iint\limits_\sigma xyz\,dS$ where σ is the portion of the unit sphere above the cone $z = \sqrt{x^2+y^2}$.

$$\begin{aligned}\iint\limits_\sigma xyz\,dS &= \iint\limits_{x^2+y^2\le\frac{1}{2}} xy\sqrt{1-x^2-y^2}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2-y^2}}\right)^2+\left(\frac{-y}{\sqrt{1-x^2-y^2}}\right)^2}\,dA\\ &= \iint\limits_{x^2+y^2\le\frac{1}{2}} xy\sqrt{1-x^2-y^2}\frac{1}{\sqrt{1-x^2-y^2}}\,dA\\ &= \iint\limits_{x^2+y^2\le\frac{1}{2}} xy\,dA\\ &= \int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} r^3\cos\theta\sin\theta\,dr\,d\theta\\ &= \left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^{\frac{1}{\sqrt{2}}}\\ &= \frac{1}{2}(0-0)(\frac{1}{16}-0) = 0.\end{aligned}$$

2. Evaluate the following integrals:

   (a) $\int_C (2xy - \sin x)\,dx + (x^2+2y)\,dy$ where C is the line segment from (0,0) to (-5,7), followed by the arc of the parabola $y = x^2+6x+12$ from $x=-5$ to $x=-3$, followed by the arc of the semicircle $x = \sqrt{18-y^2}$ from $y=3$ to $y=\sqrt{18}$.

$$\int_C \nabla(x^2y+\cos x+y^2)\cdot d\boldsymbol{r} = \int_{(0,0)}^{(0,\sqrt{18})} \nabla(x^2y+\cos x+y^2)\cdot d\boldsymbol{r} = 0^2\sqrt{18}+1+18-(0^2 0+1+0^2) = 18.$$

   (b) $\oint \langle x, y, x^2+y^2\rangle \cdot d\boldsymbol{r}$ where C is the boundary of the portion of the paraboloid $z = 1-x^2-y^2$ in the first octant.

$$\begin{aligned}\iint\limits_\sigma \operatorname{curl}\langle x, y, x^2+y^2\rangle\cdot\boldsymbol{n}\,dS &= \iint_R \langle 2y, -2x, 0\rangle\cdot\langle 2x, 2y, 1\rangle\,dA\\ &= \iint_R (4xy - 4xy + 0)\,dA = \iint_R 0\,dA = 0\end{aligned}$$

3. Evaluate the following integrals:

   (a) $\int_C (4+e^{\sqrt{x}})\,dx + (\sin y + 3x^2)\,dy$ where C is the boundary of the region enclosed by the circles $x^2+y^2=1, \quad x^2+y^2=4$ in the first quadrant.

$$\iint_R (6x-0)\,dA = \int_0^{\frac{\pi}{2}}\int_1^2 6r^2\cos\theta\,dr d\theta = [\sin\theta]_0^{\frac{\pi}{2}}\left[2r^3\right]_1^2 = (1-0)(16-2) = 14$$

   (b) $\iint\limits_\sigma (\text{curl}\,\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\rangle \cdot \boldsymbol{n}\,dS$, σ is the top half of the hemisphere $x^2+y^2+z^2=a^2$.

   Let σ_2 be the disk $x^2+y^2\le a$, $z=0$.

$$\iint\limits_\sigma \text{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot \boldsymbol{n}\,dS$$

$$= \iint\limits_{\sigma_2} \text{curl}\left\langle x^2e^{yz}, y^2e^{xz}, z^2e^{xy}\right\rangle \cdot \boldsymbol{n}\,dS$$

$$= \iint\limits_{\sigma_2} \left\langle x(z^2e^{xy}-y^2e^{xz}), y(x^2e^{yz}-z^2e^{xy}), z(y^2e^{xz}-x^2e^{yz})\right\rangle \cdot \langle 0,0,1\rangle\;dS$$

$$= \iint\limits_{\sigma_2} z(y^2e^{xz}-x^2e^{yz})\,dS$$

$$= \iint\limits_{\sigma_2} 0(y^2e^0-x^2e^0)\,dS$$

$$= \iint\limits_{\sigma_2} 0\,dS$$

$$= 0.$$

4. In the following problems, find the flux of the vector field $\boldsymbol{F}$ through the surface σ:

   (a) $\boldsymbol{F} = \langle x^2, y^2, z^2\rangle$, σ is the surface of the cube bounded by $x=0,\ y=0,\ z=0,\ x=a,\ y=a,\ z=a,\ (a>0)$.

$$\Phi = \int_0^a\int_0^a\int_0^a \text{div}\left\langle x^2,y^2,z^2\right\rangle\;dx\,dy\,dz = \int_0^a\int_0^a\int_0^a (2x+2y+2z)\,dx\,dy\,dz = 3a^4.$$

   (b) $\boldsymbol{F} = \langle xze^y, -xze^y, z\rangle$, σ is the portion of $x+y+z=1$ in the first octant, oriented downward.

$$\begin{aligned}
\Phi &= \iint_\sigma \boldsymbol{F} \cdot \boldsymbol{n}\, dS \\
&= \iint_R \langle xze^y, -xze^y, z\rangle \cdot \langle -1, -1, -1\rangle \; dA \\
&= \iint_R -xze^y + xze^y - z\, dA \\
&= \int_0^1 \int_0^{1-x} -1 + x + y\, dx\, dy \\
&= \int_0^1 (x-1)(1-x) + \frac{1}{2}(1-x)^2\, dx \\
&= \int_0^1 -\frac{(1-x)^2}{2}\, dx \\
&= \left[\frac{1}{6}(1-x)^3\right]_0^1 \\
&= -\frac{1}{6}
\end{aligned}$$

5. TRUE or FALSE? IF true, justify your answer. Otherwise, give a counterexample or a complete explanation why it is false.

(a) If $\oint_C \boldsymbol{F} \cdot d\boldsymbol{r} = 0$ for a simple closed curve C, then $\boldsymbol{F}$ is conservative.

False: See quiz 17.

(b) For every closed surface σ, and vector field $\boldsymbol{F}$ with continuous second partial derivatives, we have that $\iint_\sigma (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n}\, dS = 0$.

True: We can use the divergence theorem since σ is a closed surface, and $\boldsymbol{F}$ is "nice". Then

$$\iint_\sigma (\text{curl}\boldsymbol{F}) \cdot \boldsymbol{n}\, dS = \iiint_G \text{div}(\text{curl}\boldsymbol{F})\, dV = \iiint_G 0\, dV = 0.$$

(c) If $\nabla^2 f \; (= \nabla \cdot \nabla f) = 0$ then $\int_C f_y\, dx - f_x\, dy$ is path independent in any simply-connected region.

True: Since $0 = \nabla^2 f = f_{xx} + f_{yy}$, then $\frac{\partial}{\partial x}(-f_x) = -f_{xx} = f_{yy} = \partial \partial y f_y$. Thus, the vector field is conservative. Hence, line integrals are path independent.

(d) There is a vector field $\boldsymbol{F}$ such that $\text{curl}\boldsymbol{F} = \langle 2x, 3yz, -xz^2\rangle$.

False: Since $\text{div} \circ \text{curl} \equiv 0$ and $\text{div}\langle 2x, 3yz, -xz^2\rangle \neq 0$, then the vector field can not be the curl of another vector field.