Exam 3 Solutions

1. Evaluate the following integrals:

   (a) $\int_{-2}^{4}\int_{0}^{3}\int_{1}^{2} xz^2\,dy\,dz\,dx =$

$$\left(\int_{-2}^{4} x\,dx\right)\left(\int_{0}^{3} z^2\,dz\right)\left(\int_{1}^{2} dy\right) = \left[\frac{1}{2}x^2\right]_{-2}^{4}\cdot\left[\frac{1}{3}\right]_{0}^{3}\cdot\Big[y\Big]_{1}^{2}$$
$$= (8-2)\cdot(9-0)\cdot(2-1)$$
$$= 6\cdot 9\cdot 1 = 54.$$

   (b) $\int_{0}^{16}\int_{\sqrt{y}}^{4} e^{x^3}\,dx\,dy =$

$$\int_{0}^{4}\int_{0}^{x^2} e^{x^3}\,dy\,dx = \int_{0}^{4} x^2e^{x^3}\,dx = \left[\frac{1}{3}e^{x^3}\right]_{0}^{4} = \frac{e^{64}-1}{3}.$$

   (c) $\int_{0}^{\frac{1}{\sqrt{2}}}\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx + \int_{\frac{1}{\sqrt{2}}}^{\sqrt{2}}\int_{x}^{\sqrt{4-x^2}} \sqrt{x^2+y^2}\,dy\,dx =$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{1}^{2} \sqrt{r^2}\,r\,dr\,d\theta = \frac{\pi}{4}\left[\frac{1}{3}r^3\right]_{1}^{2} = \frac{7}{12}\pi.$$

   (d) $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz\,dy\,dx =$

$$\int_{0}^{\pi}\int_{0}^{\frac{\pi}{2}}\int_{0}^{2} \sqrt{\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \pi\Big[-\cos\phi\Big]_{0}^{\frac{\pi}{2}}\cdot\left[\frac{1}{4}\rho^4\right]_{0}^{2} = \pi\cdot 1\cdot 4 = 4\pi.$$

2. Find the volume of the solid that is in the first octant, and bounded above by the plane $x+2y+3z=6$.

$$\int_{0}^{3}\int_{0}^{6-2y}\left(2-\frac{2}{3}y-\frac{1}{3}x\right)dx\,dy = \int_{0}^{3}\left[2x-\frac{2}{3}xy-\frac{1}{6}x^2\right]_{0}^{6-2y} dy$$
$$= \int_{0}^{3}\left(12-4y-4y+\frac{4}{3}y^2-6+4y-\frac{2}{3}y^2\right)dy$$
$$= \int_{0}^{3}\left(6-4y+\frac{2}{3}y^2\right)dy$$
$$= \left[6y-2y^2+\frac{2}{9}y^3\right]_{0}^{3} = 18-18+6 = 6.$$

3. Find the surface area of the surface defined by $\mathbf{r}(u,v) = \langle u\cos v, u^2, u\sin v\rangle$, $0\le u\le\sqrt{2},\quad 0\le v\le 2\pi$.

$$\mathbf{r}_u\times\mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & 2u & \sin v \\ -u\sin v & 0 & u\cos v \end{vmatrix} = \langle 2u^2\cos v, -(u\cos^2 v + u\sin^2 v), 2u^2\sin v\rangle$$
$$= \langle 2u^2\cos v, -u, 2u^2\sin v\rangle$$

$$S = \int_0^{2\pi} \int_0^{\sqrt{2}} u\sqrt{4u^2+1}\, du\, dv = 2\pi \left[\frac{1}{12}(4u^2+1)^{\frac{3}{2}} \right]_0^{\sqrt{2}} = \frac{\pi}{6}(27-1) = \frac{13}{3}\pi.$$

4. Find the center of gravity of the hemispherical solid bounded by $z = \sqrt{a^2 - x^2 - y^2}, z = 0$ and has density at each point proportional to the distance from the origin.

$$M = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{a} k\sqrt{\rho^2}\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = 2\pi k \left[-\cos\phi \right]_0^{\frac{\pi}{2}} \left[\frac{1}{4}\rho^4 \right]_0^4 = \frac{\pi k}{2} a^4$$

$$\begin{aligned} \bar{z} &= \frac{1}{M} \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{a} k\sqrt{\rho^2}\rho^2 \sin\phi \cdot \rho\cos\phi \, d\rho\, d\phi\, d\theta \\ &= 2\pi k \frac{2}{\pi k a^4} \left[-\frac{1}{2}\cos^2\phi \right]_0^{\frac{\pi}{2}} \left[\frac{1}{5}\rho^5 \right]_0^a = \frac{4}{a^4} \cdot \frac{1}{2} \cdot \frac{1}{5} a^5 = \frac{2}{5} a \end{aligned}$$

By symmetry, $\bar{x} = \bar{y} = 0$. Thus, $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{2}{5}a)$.

5. Using the transformation $u = x - y, v = x + y$, evaluate $\iint_R \frac{x-y}{x+y}\, dA$ where R is the square with corners at $(0,2), (1,1), (2,2), (1,3)$.

$$x = \frac{u+v}{2} \qquad y = \frac{v-u}{2} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

$$\int_2^4 \int_{-2}^{0} \frac{u}{v} \cdot \frac{1}{2}\, du\, dv = \int_2^4 \frac{1}{4} \left[\frac{u^2}{v} \right]_{-2}^{0} dv = \int_2^4 -\frac{1}{v}\, dv = \left[-\ln v \right]_2^4 = -2\ln 2 + \ln 2 = -\ln 2.$$

6. Three identical cylinders with radii R intersect at the same point at right angles. Find the volume of their intersection.

   Consider the cylinders $x^2 + y^2 = R^2, x^2 + z^2 = R^2, y^2 + z^2 = R^2$. Using symmetry, we have

$$\begin{aligned} 16 \int_0^{\frac{\pi}{4}} \int_0^{R} r\sqrt{R^2 - r^2\cos^2\theta}\, dr\, d\theta &= 16 \int_0^{\frac{\pi}{4}} \left[-\frac{1}{3\cos^2\theta} \left(R^2 - r^2\cos^2\theta \right)^{\frac{3}{2}} \right]_0^R d\theta \\ &= 16 \int_0^{\frac{\pi}{4}} -\frac{1}{3\cos^2\theta} R^3 \sin^3\theta + \frac{R^3}{3\cos^2\theta}\, d\theta \\ &= \frac{16}{3} R^3 \int_0^{\frac{\pi}{4}} \left(\sec^2 - \frac{1-\cos^2\theta}{\cos^2\theta} \sin\theta \right) d\theta \\ &= \frac{16}{3} R^3 \int_0^{\frac{\pi}{4}} \left(\sec^2\theta - \frac{\sin\theta}{\cos^2\theta} + \sin\theta \right) d\theta \\ &= \frac{16}{3} R^3 \left[\tan\theta - \frac{1}{\cos\theta} - \cos\theta \right]_0^{\frac{\pi}{4}} \\ &= \frac{16}{3} R^3 \left(1 - \sqrt{2} - \frac{\sqrt{2}}{2} - 0 + 1 + 1 \right) \\ &= \frac{16}{3} R^3 \left(3 - \frac{3}{2}\sqrt{2} \right) = 8R^3 \left(2 - \sqrt{2} \right). \end{aligned}$$