Series Challenge Problems

1. Find the sum of the series
$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\frac{1}{16}+\frac{1}{18}+\frac{1}{24}+\frac{1}{27}+\cdots$$
where the terms are reciprocals of positive integers that are products of only 2's and 3's.

2. Find the interval of convergence of $\sum_{n=1}^{\infty} n^3 x^n$ and find its sum.

3. Suppose
$$\begin{aligned} u &= 1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots \\ v &= x+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\cdots \\ w &= \frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots \end{aligned}$$
Show that $u^3+v^3+w^3-3uvw=1$.

4. For $p>1$, evaluate
$$\frac{1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots}{1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+-\cdots}.$$

5. Solve the initial value problem
$$y''-2xy'-2y=0, \quad y(0)=1, \quad y'(0)=0$$

6. Let $a_0=a_1=1, \quad n(n-1)a_n=(n-1)(n-2)a_{n-1}-(n-3)a_{n-2}$.
Evaluate $\sum_{n=1}^{\infty} a_n$.

7. Show that the series of reciprocals of positive integers that do not have 0 as a digit converges, and has sum less than 90.

8. Show that the radius of convergence of the power series $\sum_{n=0}^{\infty}\frac{(pn)!}{(n!)^p}x^n$ is $\frac{1}{p^p}$ for all positive integers p.

9. Show that for all positive integers p, q, the power series $\sum_{n=0}^{\infty}\frac{(n+p)!}{n!(n+q)!}x^n$ has an infinite radius of convergence.

10. Using the Maclaurin series for xe^x, show that $\sum_{n=0}^{\infty}\frac{n+1}{n!}=2e$.

11. Verify $\int_0^1 \frac{1}{x^x}\,dx=\sum_{n=1}^{\infty}\frac{1}{n^n}$.

12. Find the sum of each of the following series:

(a) $\sum_{n=0}^{\infty} x^n, \quad |x| < 1$

(b) $\sum_{n=1}^{\infty} nx^{n-1}, \quad |x| < 1$

(c) $\sum_{n=1}^{\infty} nx^n, \quad |x| < 1$

(d) $\sum_{n=1}^{\infty} \frac{n}{2^n}$

(e) $\sum_{n=2}^{\infty} n(n-1)x^n, \quad |x| < 1$

(f) $\sum_{n=2}^{\infty} \frac{n^2 - n}{2^n}$

(g) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$