MATH 2300 - Fall 2008
Exam 3 Solutions

1. (a)

$$\sum_{k=2}^{\infty}\left(-\frac{3}{4}\right)^{k+1} = \left(-\frac{3}{4}\right)^{3}\sum_{k=2}^{\infty}\left(-\frac{3}{4}\right)^{k-2} = -\frac{27}{64}\frac{1}{1+\frac{3}{4}} = -\frac{27}{64}\frac{4}{7} = -\frac{27}{112}$$

(b)

$$\sum_{k=0}^{n}\tan^{-1}(k+1)-\tan^{-1}(k) = \sum_{k=1}^{n+1}\tan^{-1}(k)-\sum_{k=0}^{n}\tan^{-1}(k) = \tan^{-1}(n+1)-\tan^{-1}(0) = \tan^{-1}(n+1).$$

$$\sum_{k=0}^{\infty}\tan^{-1}(k+1)-\tan^{-1}(k) = \lim_{n\to\infty}\sum_{k=0}^{n}\tan^{-1}(k+1)-\tan^{-1}(k) = \lim_{n\to\infty}\tan^{-1}(n+1) = \frac{\pi}{2}.$$

2. (a) $\displaystyle\lim_{k\to\infty}\frac{1}{\pi^{1/k}} = 1 \neq 0$. So, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{\pi^{1/k}}$ diverges by the Divergence Test.

(b) Since $\{k\}$ and $\{\ln k\}$ are monotone increasing to infinity, then $\{\frac{1}{k\ln k}\}$ is monotone decreasing to zero. So, by the Alternating Series Test, $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$ converges.

Also, x and $\ln x$ are continuous functions and nonzero for $x > 0$. So, $\frac{1}{x\ln x}$ is also continuous. Furthermore, it is decreasing since x and $\ln x$ are increasing. Note that

$$\int_2^{\infty}\frac{1}{x\ln x}\,dx = \int_{\ln 2}^{\infty}\frac{1}{u}\,du = \lim_{a\to\infty}[\ln u]_{\ln 2}^{a} = \lim_{a\to\infty}\ln a - \ln(\ln 2) = \infty.$$

So, by the Integral Test, $\displaystyle\sum_{k=2}^{\infty}\frac{1}{k\ln k}$ diverges. Thus, $\displaystyle\sum_{k=2}^{\infty}\frac{(-1)^k}{k\ln k}$ converges conditionally.

3. (a)

$$\sum_{k=1}^{\infty}\frac{4+(-1)^k}{k^5} \leq \sum_{k=1}^{\infty}\frac{5}{k^5}$$

The series converges absolutely by a comparison to a convergent p-series.

(b)

$$\lim_{k\to\infty}\frac{\frac{1}{2k+\ln k}}{\frac{1}{k}} = \lim_{k\to\infty}\frac{1}{2+\frac{\ln k}{k}} = \frac{1}{2+e^{\lim_{k\to\infty}\frac{\ln k}{k}}} \overset{\text{L'H}}{=} \frac{1}{2+e^{\lim_{k\to\infty}\frac{1/k}{1}}} = \frac{1}{2+e^0} = \frac{1}{3}.$$

Since $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k}$ is a divergent p-series, we have by the Limit Comparison Test, that $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k+\ln k}$ also diverges.

OR: $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k+\ln k} \geq \sum_{k=1}^{\infty}\frac{1}{2k\ln k+k\ln k} = \sum_{k=1}^{\infty}\frac{1}{3k\ln k}$, which diverges by 2(b). So, $\displaystyle\sum_{k=1}^{\infty}\frac{1}{2k+\ln k}$ diverges by the Comparison test.

4. (a)

$$\sum_{k=7}^{\infty} \frac{1}{(2k-5)^{\frac{5}{2}}} \leq \sum_{k=7}^{\infty} \frac{1}{(2k-k)^{\frac{5}{2}}} = \sum_{k=7}^{\infty} \frac{1}{k^{\frac{5}{2}}}$$

The series converges absolutely by a comparison to a convergent p-series.

(b)

$$\lim_{k\to\infty} \frac{(3k+3)!}{3^{k+1}} \cdot \frac{3^k}{(3k)!} = \lim_{k\to\infty} \frac{(3k+3)(3k+2)(3k+1)}{3} = \lim_{k\to\infty} (k+1)(3k+2)(3k+1) = \infty.$$

So, the series diverges by the Ratio Test.

5.

$$\sum_{k=2}^{\infty} \frac{3}{(2k)^2} = -\frac{3}{4} + \sum_{k=1}^{\infty} \frac{3}{4k^2} = -\frac{3}{4} + \frac{3}{4}\sum_{k=1}^{\infty} \frac{1}{k^2} = -\frac{3}{4} + \frac{3}{4} \cdot \frac{\pi^2}{6} = \frac{3}{4}\left(\frac{\pi^2-6}{6}\right) = \frac{\pi^2-6}{8}.$$

6.

$$\begin{aligned} f^{(k)}(x) &= (-1)^k e^{-x} \\ f^{(k)}(0) &= (-1)^k \\ p_n(x) &= \sum_{k=0}^{n} \frac{(-1)^k}{k!} x^k \\ &= \sum_{k=0}^{n} \frac{(-x)^k}{k!}. \end{aligned}$$