MATH 2300 - Fall 2008
Exam 2 Solutions

1.

$$\int \frac{3x^3}{\sqrt{1-x^2}}\,dx$$
$$x = \sin\theta \qquad dx = \cos\theta\,d\theta$$

$$\begin{aligned}
\int \frac{3x^3}{\sqrt{1-x^2}}\,dx &= \int \frac{3\sin^3\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta \\
&= \int \frac{3\sin^3\theta\cos\theta}{\cos\theta}\,d\theta \\
&= \int 3\sin^3\theta\,d\theta \\
&= 3\int \left(1-\cos^2\theta\right)\sin\theta\,d\theta \\
&= 3\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right) + C \\
&= -3\sqrt{1-x^2} + \left(1-x^2\right)^{\frac{3}{2}} + C \\
&= \left(-x^2-2\right)\sqrt{1-x^2} + C.
\end{aligned}$$

2.

$$\int \frac{x-10}{2x^2-5x-3}\,dx$$

$$\begin{aligned}
\frac{x-10}{(2x+1)(x-3)} &= \frac{A}{2x+1} + \frac{B}{x-3} \\
x-10 &= A(x-3) + B(2x+1) \\
x=3: \quad -7 &= 7B \\
-1 &= B \\
x=0: \quad -10 &= -3A-1 \\
-9 &= -3A \\
3 &= A \\
\int \frac{x-10}{2x^2-5x-3}\,dx &= \int \frac{3}{2x+1} + \frac{-1}{x-3}\,dx \\
&= \frac{3}{2}\ln|2x+1| - \ln|x-3| + C.
\end{aligned}$$

3. $\int_{-2}^{2} \frac{1}{(x-1)^2}\,dx$

$$\begin{aligned}
\int_{1}^{2} \frac{1}{(x-1)^2}\,dx &= \lim_{a\to 1^+} \int_{a}^{2} \frac{1}{(x-1)^2}\,dx \\
&= \lim_{a\to 1^+} \left[-\frac{1}{x-1}\right]_a^2 \\
&= \lim_{a\to 1^+} \left(-1+\frac{1}{a-1}\right) \\
&= -1+\infty \\
&= \infty.
\end{aligned}$$

So, $\int_{1}^{2} \frac{1}{(x-1)^2}\,dx$ diverges.

Thus, $\int_{-2}^{2} \frac{1}{(x-1)^2}\,dx = \int_{-2}^{1} \frac{1}{(x-1)^2}\,dx + \int_{1}^{2} \frac{1}{(x-1)^2}\,dx$ also diverges.

4. (a)

$$\begin{aligned}
y' - x &= -5y \\
y' + 5y &= x \\
\mu(x) &= e^{\int 5\,dx} = e^{5x} \\
y(x) &= e^{-5x} \int x e^{5x}\,dx \\
&= e^{-5x} \left(\frac{1}{5} x e^{5x} - \int \frac{1}{5} e^{5x}\,dx\right) \\
&= e^{-5x} \left(\frac{1}{5} x e^{5x} - \frac{1}{25} e^{5x} + C\right) \\
&= \frac{1}{5}x - \frac{1}{25} + Ce^{-5x} \\
1 = y\left(\frac{1}{5}\right) &= \frac{1}{25} - \frac{1}{25} + Ce^{-1} \\
e &= C \\
y(x) &= \frac{1}{5}x - \frac{1}{25} + ee^{-5x} \\
&= -\frac{1}{25}(-5x+1) + e^{-5x+1}.
\end{aligned}$$

(b)

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x^2}{1+y^2} \\
\int 1+y^2\,dy &= \int x^2\,dx \\
y + \frac{1}{3}y^3 &= \frac{1}{3}x^3 + C \\
y^3 + 3y &= x^3 + C.
\end{aligned}$$

5. (a)

$$\begin{aligned} y'' + y' - 6y &= 0 \\ m^2 + m - 6 &= 0 \\ (m+3)(m-2) &= 0 \end{aligned}$$

$$y(x) = c_1 e^{-3x} + c_2 e^{2x}.$$

(b)

$$\begin{aligned} y'' - 6y' + 13y &= 0 \\ m^2 - 6m + 13 &= 0 \end{aligned}$$

$$\begin{aligned} m &= \frac{6 \pm \sqrt{36-52}}{2} \\ &= \frac{6 \pm \sqrt{-16}}{2} \\ &= \frac{6 \pm 4i}{2} \\ &= 3 \pm 2i \end{aligned}$$

$$y(x) = e^{3x}\left(c_1 \sin(2x) + c_2 \cos(2x)\right).$$

6. (a)

$$\{a_n\}_{n=1}^{\infty} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots = \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{\infty}.$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2^{n-1}} = \lim_{n\to\infty} \left(\frac{1}{2}\right)^{n-1} = 0 \ \text{ since } \left|\frac{1}{2}\right| < 1.$$

(b)

$$\lim_{n\to\infty} \frac{3n^2+1}{6n^2+5} = \lim_{n\to\infty} \frac{3 + \frac{1}{n^2}}{6 + \frac{5}{n^2}} = \frac{3+0}{6+0} = \frac{1}{2}.$$

7.

$$\begin{aligned} a_n &= \frac{2^{n+1}}{(n+1)!} \\ \frac{a_{n+1}}{a_n} &= \frac{2^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{2^{n+1}} \\ &= \frac{2}{n+2} \\ &< \frac{2}{0+2} \\ &= 1 \text{ for all } n \geq 1. \end{aligned}$$

Thus, $\left\{\frac{2^{n+1}}{(n+1)!}\right\}_{n=1}^{\infty}$ is strictly decreasing, and so, not strictly increasing.