MATH 2300 - Fall 2008
Exam 1 Solutions

1. (a)

$$\frac{d}{dx}\left[\ln(\cosh x)\right] = \frac{1}{\cosh x} \cdot \sinh x = \frac{\sinh x}{\cosh x} = \tanh x.$$

   (b)

$$\begin{aligned}
\int e^x \cosh x \, dx &= \int e^x \left(\frac{e^x + e^{-x}}{2} \right) dx \\
&= \frac{1}{2} \int e^{2x} + 1 \, dx \\
&= \frac{1}{2} \left(\frac{1}{2} e^{2x} + x \right) + C \\
&= \frac{1}{4} e^{2x} + \frac{1}{2} x + C.
\end{aligned}$$

2.

$$u = \ln x \qquad\qquad v = \frac{1}{3} x^3$$

$$du = \frac{1}{x} \, dx \qquad\qquad dv = x^2 \, dx$$

$$\begin{aligned}
\int x^2 \ln x \, dx &= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^3 \cdot \frac{1}{x} \, dx \\
&= \frac{1}{3} x^3 \ln x - \int \frac{1}{3} x^2 \, dx \\
&= \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.
\end{aligned}$$

3.

$$\begin{aligned}
\int \sin^2 x \cos^3 x \, dx &= \int \sin^2 x \left(1 - \sin^2 x\right) \cos x \, dx \\
&= \int u^2 \left(1 - u^2\right) \, du \\
&= \int u^2 - u^4 \, du \\
&= \frac{1}{3} u^3 - \frac{1}{5} u^5 + C \\
&= \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C.
\end{aligned}$$

4.

$$\begin{aligned}
\int_0^{\frac{\sqrt{\pi}}{2}} x^3 \sin(x^2)\,dx &= \int_0^{\frac{\sqrt{\pi}}{2}} \frac{1}{2}x^2 \sin(x^2)2x\,dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{2} y \sin y\,dy \\
u &= \frac{1}{2}y & v &= -\cos y \\
du &= \frac{1}{2}dy & dv &= \sin y\,dy \\
&= \left[-\frac{1}{2}y\cos y\right]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \frac{1}{2}\cos y\,dy \\
&= \left[-\frac{1}{2}y\cos y + \frac{1}{2}\sin y\right]_0^{\frac{\pi}{4}} \\
&= -\frac{1}{2}\cdot\frac{\pi}{4}\cdot\frac{\sqrt{2}}{2} + \frac{1}{2}\cdot\frac{\sqrt{2}}{2} + 0 - 0 \\
&= \frac{\sqrt{2}}{16}(4-\pi).
\end{aligned}$$

5.

$$f(x) = \int_{\sin x}^{\cos x} e^{t^2}\,dt$$

$$f\left(\frac{\pi}{4}\right) = \int_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} e^{t^2}\,dt = 0$$

$$f'(x) = e^{\cos^2 x}(-\sin x) - e^{\sin^2 x}(\cos x)$$

$$f'\left(\frac{\pi}{6}\right) = e^{\left(\frac{\sqrt{3}}{2}\right)^2}\left(-\frac{1}{2}\right) - e^{\left(\frac{1}{2}\right)^2}\left(\frac{\sqrt{3}}{2}\right) = -\frac{1}{2}e^{\frac{3}{4}} - \frac{\sqrt{3}}{2}e^{\frac{1}{4}} = -\frac{e^{\frac{1}{4}}}{2}\left(\sqrt{e}+\sqrt{3}\right)$$

$$f\left(\frac{\pi}{4}\right) + f'\left(\frac{\pi}{6}\right) = 0 + -\frac{e^{\frac{1}{4}}}{2}\left(\sqrt{e}+\sqrt{3}\right) = -\frac{e^{\frac{1}{4}}}{2}\left(\sqrt{e}+\sqrt{3}\right).$$

6.

$$\begin{aligned}
L &= \int_{\frac{2\sqrt{3}}{3}}^{\frac{4\sqrt{2}}{3}} \sqrt{\left(\frac{d}{dt}\left[t^2\right]\right)^2 + \left(\frac{d}{dt}\left[t^3+1\right]\right)^2}\, dt \\
&= \int_{\frac{2\sqrt{3}}{3}}^{\frac{4\sqrt{2}}{3}} \sqrt{4t^2 + 9t^4}\, dt \\
&= \int_{\frac{2\sqrt{3}}{3}}^{\frac{4\sqrt{2}}{3}} t\sqrt{4 + 9t^2}\, dt \\
&= \int_{16}^{36} \frac{1}{18}\sqrt{u}\, du \\
&= \left[\frac{1}{27}u^{\frac{3}{2}}\right]_{16}^{36} \\
&= \frac{1}{27}\left(6^3 - 4^3\right) \\
&= \frac{8}{27}\left(3^3 - 2^3\right) \\
&= \frac{152}{27}.
\end{aligned}$$

7. (a)

$$\begin{aligned}
\int \sec x\, dx &= \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)\, dx \\
&= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\
&= \int \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x}\, dx \\
&= \ln|\sec x + \tan x| + C.
\end{aligned}$$

(b)

$$
\begin{aligned}
L &= \int_0^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{d}{dx}\left[\ln(\cos x)\right]\right)^2}\, dx \\
&= \int_0^{\frac{\pi}{3}} \sqrt{1 + (-\tan x)^2}\, dx \\
&= \int_0^{\frac{\pi}{3}} \sqrt{1 + \tan^2 x}\, dx \\
&= \int_0^{\frac{\pi}{3}} \sqrt{\sec^2 x}\, dx \\
&= \int_0^{\frac{\pi}{3}} |\sec x|\, dx \\
&= \int_0^{\frac{\pi}{3}} \sec x\, dx \\
&= \left[\ln|\sec x + \tan x|\right]_0^{\frac{\pi}{3}} \\
&= \ln\left(2 + \sqrt{3}\right) - \ln(1 + 0) \\
&= \ln\left(2 + \sqrt{3}\right).
\end{aligned}
$$

8.

$$
\begin{aligned}
u &= \tan^{-1}(2x) & v &= x \\
du &= \frac{2}{1 + 4x^2}\, dx & dv &= dx
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \int f'(x)\, dx \\
&= \int \tan^{-1}(2x)\, dx \\
&= x\tan^{-1}(2x) - \int \frac{2x}{1 + 4x^2}\, dx \\
&= x\tan^{-1}(2x) - \int \frac{1}{4y}\, dy \\
&= x\tan^{-1}(2x) - \frac{1}{4}\ln|y| + C \\
&= x\tan^{-1}(2x) - \frac{1}{4}\ln\left(1 + 4x^2\right) + C.
\end{aligned}
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$$
f\left(\frac{1}{2}\right) = -\frac{1}{4}\ln 2 \Rightarrow
$$

$$
\begin{aligned}
-\frac{1}{4}\ln 2 &= \frac{1}{2}\tan^{-1}(1) - \frac{1}{4}\ln\left(1 + 4\left(\frac{1}{2}\right)^2\right) + C \\
-\frac{1}{4}\ln 2 &= \frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\ln 2 + C \\
0 &= \frac{\pi}{8} + C \\
-\frac{\pi}{8} &= C
\end{aligned}
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$$
f(x) = x\tan^{-1}(2x) - \frac{1}{4}\ln\left(1 + 4x^2\right) - \frac{\pi}{8}.
$$