2. The axioms of set theory

Before introducing any set-theoretic axioms at all, we can introduce some more abbreviations.

$x \subseteq y$ abbreviates $\forall z \left( z \in x \rightarrow z \in y \right)$.

$x \subset y$ abbreviates $x \subseteq y \land x \neq y$.

For $x \subseteq y$ we say that $x$ is included or contained in $y$, or that $x$ is a subset of $y$. Then $x \subset y$ means proper inclusion, containment, or subset.

Now we introduce the axioms of ZFC set theory. We give both a formal and informal description of them. The informal versions will suffice for much of these notes.

**Axiom 1.** (Extensionality) If two sets have the same members, then they are equal. Formally:

$$\forall x \forall y \left[ \forall z \left( z \in x \leftrightarrow z \in y \right) \rightarrow x = y \right].$$

Note that the other implication here holds on the basis of logic.

**Axiom 2.** (Comprehension) Given any set $z$ and any property $\varphi$, there is a subset of $z$ consisting of those elements of $z$ with the property $\varphi$.

Formally, for any formula $\varphi$ with free variables among $x, z, w_1, \ldots, w_n$ we have an axiom

$$\forall z \forall w_1 \ldots \forall w_n \exists y \forall x \left( x \in y \leftrightarrow x \in z \land \varphi \right).$$

Note that the variable $y$ is not free in $\varphi$.

From these first two axioms the existence of a set with no members, the empty set $\emptyset$, follows:

**Proposition 2.1.** There is a unique set with no members.

**Proof.** On the basis of logic, there is at least one set $z$. By the comprehension axiom, let $y$ be a set such that $\forall x \left( x \in y \leftrightarrow x \in z \land x \neq x \right)$. Thus $y$ does not have any elements. By the extensionality axiom, such a set $y$ is unique. □

In general, the set asserted to exist in the comprehension axiom is unique; we denote it by $\{ x \in z : \varphi \}$. We sometimes write $\{ x : \varphi \}$ if a suitable $z$ is evident.

**Axiom 3.** (Pairing) For any sets $x, y$ there is a set which has them as members (possibly along with other sets). Formally:

$$\forall x \forall y \exists z \left( x \in z \land y \in z \right).$$

The unordered pair $\{ x, y \}$ is by definition the set $\{ u \in z : u = x \lor u = y \}$, where $z$ is as in the pairing axiom. The definition does not depend on the particular such $z$ that is chosen. This same remark can be made for several other definitions below. We define the singleton $\{ x \}$ to be $\{ x, x \}$.

**Axiom 4.** (Union) For any family $\mathcal{A}$ of sets, we can form a new set $A$ which has as elements all elements which are in at least one member of $\mathcal{A}$ (maybe $A$ has even more elements). Formally:

$$\forall \mathcal{A} \exists A \forall X \forall Y \left( x \in Y \land Y \in \mathcal{A} \rightarrow x \in A \right).$$
With $A$ as in this axiom, we define $\bigcup A = \{ x \in A : \exists Y \in A (x \in Y) \}$. We call $\bigcup A$ the union of $A$. Also, let $x \cup y = \bigcup \{ x, y \}$. This is the union of $x$ and $y$.

**Axiom 5.** (Power set) For any set $x$, there is a set which has as elements all subsets of $x$, and again possibly has more elements. Formally:

$$\forall x \exists y \forall z (z \subseteq x \rightarrow z \in y).$$

**Axiom 6.** (Infinity) There is a set which intuitively has infinitely many elements:

$$\exists x[\emptyset \in x \land \forall y \in x (y \cup \{ y \} \in x)].$$

If we take the smallest set $x$ with these properties we get the natural numbers, as we will see later.

**Axiom 7.** (Replacement) If a function has domain a set, then its range is also a set. Here we use the intuitive notion of a function. Later we define the rigorous notion of a function. The present intuitive notion is more general, however; it is expressed rigorously as a formula with a function-like property. The rigorous version of this axiom runs as follows.

For each formula with free variables among $x, y, A, w_1, \ldots, w_n$, the following is an axiom.

$$\forall A \forall w_1 \ldots \forall w_n [\forall x \in A !y \varphi \rightarrow \exists Y \forall x \in A \exists y \in Y \varphi].$$

For the next axiom, we need another definition. For any sets $x, y$, let $x \cap y = \{ z \in x : z \in y \}$. This is the intersection of $x$ and $y$.

**Axiom 8.** (Foundation) Every nonempty set $x$ has a member $y$ which has no elements in common with $x$. This is a somewhat mysterious axiom which rules out such anti-intuitive situations as $a \in a$ or $a \in b \in a$.

$$\forall x [x \neq \emptyset \rightarrow \exists y \in x (x \cap y = \emptyset)].$$

**Axiom 9.** (Choice) This axiom will be discussed carefully later; it allows one to pick out elements from each of an infinite family of sets. A convenient form of the axiom to start with is as follows. For any family $A$ of nonempty sets such that no two members of $A$ have an element in common, there is a set $B$ having exactly one element in common with each member of $A$.

$$\forall A [\forall x \in A (x \neq \emptyset) \land \forall x \in A \forall y \in A (x \neq y \rightarrow x \cap y = \emptyset)] \rightarrow \exists B \forall x \in A !y (y \in x \land y \in B).$$