Solutions to exercises in Chapter 5

E5.1 Give an estimate for the size of \( gn(\varphi) \), where \( \varphi \) is the Peano Postulate (P1).

(P1) is the formula \( \forall v_0 \forall v_1 [Sv_0 = Sv_1 \rightarrow v_0 = v_1] \), or as a sequence

\[
\langle 4, 5, 4, 10, 2, 3, 6, 5, 6, 10, 3, 5, 10 \rangle.
\]

Hence

\[
\begin{align*}
gn(\varphi) &= p_0^4 \cdot p_1^5 \cdot p_2^2 \cdot p_3^{10} \cdot p_4^2 \cdot p_5^3 \cdot p_6^2 \cdot p_7^5 \cdot p_8^4 \cdot p_9^{10} \cdot p_{10}^3 \cdot p_{11}^5 \cdot p_{12}^1 \\
&= 2^4 \cdot 3^5 \cdot 5^4 \cdot 7^{10} \cdot 11^2 \cdot 13^3 \cdot 17^6 \cdot 19^5 \cdot 23^6 \cdot 29^{10} \cdot 31^3 \cdot 37^5 \cdot 41^{10} \\
&\approx 1.9 \cdot 10^{85}.
\end{align*}
\]

E5.2 Describe \( G(gn(v_0 = v_0)) \) and express it as a product of primes.

For brevity let \( m = gn(v_0 = v_0) \); recall from the notes that \( m = 6,075,000 \). Now by definition, \( G(gn(v_0 = v_0)) = gn(\text{Subff}_{m}(v_0 = v_0)) \); thus \( G(gn(v_0 = v_0)) \) is the Gödel number of the formula

\[
\forall v_m [v_m = \overline{m} \rightarrow \forall v_0 [v_0 = v_m \rightarrow v_0 = v_0]].
\]

As an actual sequence of integers, this formula is

\[
\langle 4, 5(m + 1), 2, 3, 5(m + 1), 6(m \text{ times}), 8, 4, 5, 2, 3, 5, 5(m + 1), 3, 5, 5 \rangle.
\]

Hence \( G(gn(v_0 = v_0)) \) is the number

\[
p_0^4 \cdot p_1^{5(m+1)} \cdot p_2^2 \cdot p_3^3 \cdot p_4^{5(m+1)} \cdot \prod_{i<m} p_{5+i}^6 \cdot p_{5+m}^8 \cdot p_{5+m+1}^4 \\
\cdot p_{5+m+2}^5 \cdot p_{5+m+3}^2 \cdot p_{5+m+4}^3 \cdot p_{5+m+5}^5 \cdot p_{5+m+6}^5 \cdot p_{5+m+7}^3 \cdot p_{5+m+8}^5 \cdot p_{5+m+9}^5
\]

Suppose that \( \Gamma \) is a set of sentences containing \( \mathbf{P}' \). A formula \( \rho \) with at most \( v_0 \) free is a \( \Gamma \)-provability condition iff for any sentence \( \varphi \), \( \Gamma \vdash \varphi \) iff \( \Gamma \vdash \rho(gn(\varphi)) \).

E5.3 Suppose that \( \Gamma \) is a set of sentences containing \( \mathbf{P}' \), and \( \overline{M} \overset{\text{def}}{=} (\omega, S, 0, +, \cdot) \) is a model of \( \Gamma \). Let \( \chi \) be as in the proof of Gödel’s incompleteness theorem, and let \( \pi \) be the formula \( \exists v_1 \chi \). Prove that \( \pi \) is a \( \Gamma \)-provability condition.

Let \( \varphi \) be a sentence. First suppose that \( \Gamma \vdash \varphi \). Let \( \Phi \) be a \( \Gamma \)-proof with last entry \( \varphi \). Thus \( (gn(\varphi), gn_1(\Phi)) \in \text{Prf}_\Gamma \), so \( \Gamma \vdash \chi(g(\varphi), g_2(\Phi)) \). Hence \( \Gamma \vdash \exists v_1 \chi(gn(\varphi)) \), so also \( \Gamma \vdash \exists v_1 \chi(gn(\varphi)) \).

Second suppose that \( \Gamma \vdash \exists v_1 \chi(gn(\varphi)) \). Then \( \overline{M} \models \exists v_1 \chi(gn(\varphi)) \), so we can choose \( m \in \omega \) such that \( \overline{M} \models \chi(gn(\varphi))[m, m] \). We claim that \( (gn(\varphi), m) \in \text{Prf}_\Gamma \). Otherwise we get \( \overline{P} \models \neg \chi(gn(\varphi), \overline{m}) \), hence \( \overline{M} \models \neg \chi(gn(\varphi))[m, m] \), contradiction. This proves the claim. Let \( \Phi \) be a \( \Gamma \)-proof with last entry \( \varphi \) such that \( m = gn_1(\Phi) \). Hence \( \Gamma \vdash \varphi \).
Suppose that \( \Gamma \) is a set of sentences containing \( P' \), and \( M \defeq (\omega, S, 0, +, \cdot) \) is a model of \( \Gamma \). Suppose that \( \rho \) is a \( \Gamma \)-provability condition. Apply the fixed point theorem to get a sentence \( \psi \) such that \( P \vdash \psi \leftrightarrow \neg \rho(gn(\psi)) \), as in the proof of Gödel's incompleteness theorem. Prove that \( \not(\Gamma \vdash \psi) \) and \( \not(\Gamma \vdash \neg \psi) \).

Suppose that \( \Gamma \vdash \psi \). Then \( \Gamma \vdash \rho(gn(\psi)) \) since \( \rho \) is a provability condition, but also \( \Gamma \vdash \neg \rho(gn(\psi)) \) by the choice of \( \psi \). So \( \Gamma \) is inconsistent. This contradicts the assumption that \( M \) is a model of \( \Gamma \). Hence \( \not(\Gamma \vdash \psi) \). Suppose that \( \Gamma \vdash \neg \psi \). Then by the choice of \( \psi \), \( \Gamma \vdash \rho(gn(\psi)) \). It follows that \( \Gamma \vdash \psi \) since \( \rho \) is a provability condition, contradiction.

Let \( \chi \) be as in the proof of Gödel's incompleteness theorem, and let \( \pi \) be the formula \( \exists v_1 \chi \). The following can be shown for \( \chi \).

(i) For any sentences \( \varphi \) and \( \psi \),
\[
\Gamma \vdash \pi(gn(\varphi \to \psi)) \to (\pi(gn(\varphi)) \to \pi(gn(\psi))).
\]

(“\( \Gamma \) proves that if \( \varphi \to \psi \) is provable, then from the provability of \( \varphi \) it follows that \( \psi \) is provable”)

(ii) For any sentence \( \varphi \),
\[
\Gamma \vdash \pi(gn(\varphi)) \to \pi(gn(\pi(gn(\varphi)))).
\]

(“\( \Gamma \) proves that if \( \varphi \) is provable, then it is provable that \( \varphi \) is provable.”)

By the fixed point theorem, let \( \psi \) be a sentence such that \( \Gamma \vdash \psi \leftrightarrow \pi(gn(\psi)) \). Note that \( \psi \) says “I am provable”. By the fixed point theorem again, let \( \theta \) be a sentence such that \( \Gamma \vdash \theta \leftrightarrow (\pi(gn(\theta)) \to \psi) \). Thus \( \theta \) says “If I am provable, then \( \psi \) holds.”

Show that if \( \Gamma \vdash \theta \), then \( \Gamma \vdash \psi \).

Let \( \Phi \) be a \( \Gamma \)-proof with last entry \( \theta \). Thus \( (gn(\theta), gn_1(\Phi)) \in \text{Prf}_\Gamma \), and it follows that \( P \vdash \chi(gn(\theta), gn_1(\Phi)) \), hence \( \Gamma \vdash \pi(gn(\theta)) \). Since also \( \Gamma \vdash \theta \), it follows that \( \Gamma \vdash \psi \).

(Continuing E5.5.) Show that \( \Gamma \vdash \pi(gn(\theta)) \to \pi(gn(\psi)) \).

By the choice of \( \theta \) we have \( \Gamma \vdash \theta \to (\pi(gn(\theta)) \to \psi) \), and then by a tautology we have \( \Gamma \vdash \pi(gn(\theta)) \to (\theta \to \psi) \). By exercise E5.3 we then get
\[
\Gamma \vdash \pi(gn(\pi(gn(\theta)) \to (\theta \to \psi))),
\]
and E5.5(i) gives
\[
\Gamma \vdash \pi(gn(\pi(gn(\pi(gn(\theta))))) \to \pi(gn(\theta \to \psi))).
\]

Another instance of E5.5(i) is
\[
\Gamma \vdash \pi(gn(\theta \to \psi)) \to (\pi(gn(\theta)) \to \pi(gn(\psi))).
\]
Two instances of (ii) are

\begin{align}
\Gamma \vdash \pi(gn(\theta)) \rightarrow \pi(gn(\pi(gn(\theta)))) & \quad \text{and} \\
\Gamma \vdash \pi(gn(\pi(gn(\theta)))) \rightarrow \pi(gn(\pi(gn(\pi(gn(\theta)))))).
\end{align}

Now a simple tautology gives \( \Gamma \vdash \varphi(gn(\theta)) \rightarrow \pi(gn(\theta \rightarrow \psi)) \), using (3), (4), (1). The more complicated tautology

\[ [S_0 \rightarrow (S_1 \rightarrow S_2)] \rightarrow [(S_1 \rightarrow S_0) \rightarrow (S_1 \rightarrow S_2)] \]

gives the desired conclusion. [Substitute \( \pi(g(\theta \rightarrow \psi)) \) for \( S_0 \), \( \pi(gn(\theta)) \) for \( S_1 \), and \( \pi(gn(\psi)) \) for \( S_2 \)]

**E5.7** (Continuing E5.5.) Show that \( \Gamma \vdash \psi \).

By the choice of \( \theta \) we have \( \Gamma \vdash (\pi(g(\theta)) \rightarrow \psi) \rightarrow \theta \). Using the definition of \( \psi \) and a tautology, we get \( \Gamma \vdash (\pi(gn(\theta)) \rightarrow \pi(gn(\psi))) \rightarrow \theta \). Then by exercise 5.6 it follows that \( \Gamma \vdash \theta \). Hence by exercise 5.5, \( \Gamma \vdash \psi \).

**E5.8** Assume that \( \Gamma \vdash \neg (\overline{m} = \overline{n}) \) for all distinct \( m, n \in \omega \). Let \( \chi \) be as in the proof of Gödel's incompleteness theorem, and let \( \theta \) be the sentence \( \forall v_0 (v_0 = v_0) \). Prove that the following formula \( \rho(v_0) \) is a provability condition:

\[ v_0 = gn(\theta) \lor \exists v_1 \chi(v_0, v_1) \]

Let \( \varphi \) be any sentence. If \( \Gamma \vdash \varphi \), then \( \Gamma \vdash \exists v_1 \chi(gn(\varphi), v_1) \) by exercise E5.3, and so clearly \( \Gamma \vdash \rho(gn(\varphi)) \). Suppose that \( \Gamma \vdash \rho(gn(\varphi)) \). If \( \varphi = \theta \), obviously \( \Gamma \vdash \varphi \). If \( \varphi \neq \theta \), then \( gn(\varphi) \neq gn(\theta) \) and hence by assumption \( \Gamma \vdash \neg (gn(\varphi) = gn(\theta)) \), and a tautology gives \( \Gamma \vdash \exists v_1 \chi(gn(\varphi), v_1) \). Hence by exercise E5.3, \( \Gamma \vdash \varphi \).

**E5.9** (Continuing E5.8) Prove that \( \Gamma \vdash \theta \leftrightarrow \rho(gn(\theta)) \) (so that \( \theta \) asserts its own provability with respect to this condition).

In fact, clearly \( \Gamma \vdash \theta \), and also \( \Gamma \vdash \rho(g(\theta)) \), so the conclusion follows.

**E5.10** (Continuing E5.8) Prove that \( \Gamma \vdash \theta \).

This is clear.

**E5.11** Assume that \( \Gamma \vdash \neg (\overline{m} = \overline{n}) \) for all distinct \( m, n \in \omega \). Let \( \chi \) be an in the proof of Gödel's incompleteness theorem, and let \( \theta \) be the sentence \( \neg \forall v_0 (v_0 = v_0) \). Prove that the following formula \( \rho(v_0) \) is a provability condition:

\[ \neg (v_0 = gn(\theta)) \land \exists v_1 \chi(v_0, v_1) \]

Let \( \varphi \) be a sentence. Suppose that \( \Gamma \vdash \varphi \). Then by exercise E5.3, \( \Gamma \vdash \exists v_1 \chi(g(\varphi), v_1) \). Now \( \varphi \neq \theta \), since \( \Gamma \vdash \neg \theta \), and \( \Gamma \) is consistent since \( M \) is a model of it. Hence \( gn(\varphi) \neq gn(\theta) \). So by assumption \( \Gamma \vdash \neg (gn(\varphi) = gn(\theta)) \). Hence \( \Gamma \vdash \rho(gn(\varphi)) \).
Suppose that $\Gamma \vdash \rho(gn(\varphi))$. Then also $\Gamma \vdash \exists v_1 x(gn(\varphi), v_1)$, so by exercise E5.3, $\Gamma \vdash \varphi$.

(E5.12) (Continuing E5.11) Show that $\Gamma \vdash \theta \leftrightarrow \rho(g(\theta))$, so that $\theta$ asserts its own provability.

In fact, clearly $\Gamma \vdash \neg \theta$ and also $\Gamma \vdash \neg \rho(g(\theta))$, so the exercise follows.

(E5.13) (Continuing E5.11) Show that $\Gamma \vdash \neg \theta$.

This is clear.