Solutions to exercises in Chapter 4

E4.1 Suppose that $\Gamma \vdash \varphi \rightarrow \psi$, $\Gamma \vdash \varphi \rightarrow \neg \psi$, and $\Gamma \vdash \neg \varphi \rightarrow \varphi$. Prove that $\Gamma$ is inconsistent.

The formula $(\neg \varphi \rightarrow \varphi) \rightarrow \varphi$ is a tautology. Hence by Lemma 3.3, $\Gamma \vdash (\neg \varphi \rightarrow \varphi) \rightarrow \varphi$.

Since also $\Gamma \vdash \neg \varphi \rightarrow \varphi$, it follows that $\Gamma \vdash \varphi$. Hence $\Gamma \vdash \psi$ and $\Gamma \vdash \neg \psi$. Hence by Lemma 4.1, $\Gamma$ is inconsistent.

E4.2 Let $L$ be a language with just one non-logical constant, a binary relation symbol $R$.

Let $\Gamma$ consist of all sentences of the form $\exists v_1 \forall v_0 [Rv_0v_1 \leftrightarrow \varphi]$ with $\varphi$ a formula with only $v_0$ free. Show that $\Gamma$ is inconsistent. Hint: take $\varphi$ to be $\neg Rv_0v_0$.

By Theorem 3.27 we have

1. $\Gamma \vdash \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0] \rightarrow [Rv_1v_1 \leftrightarrow \neg Rv_1v_1]$.

Now $[Rv_1v_1 \leftrightarrow \neg Rv_1v_1] \rightarrow (v_0 = v_0)$ is a tautology, so from (1) we obtain

$\Gamma \vdash \exists v_1 \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0] \rightarrow (v_0 = v_0)$;

then generalization gives

$\Gamma \vdash \exists v_1 \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0] \rightarrow (v_0 = v_0)]$.

Then by Proposition 3.39 we get

$\Gamma \vdash \exists v_1 \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0] \rightarrow (v_0 = v_0)$.

But the hypothesis here is a member of $\Gamma$, so we get $\Gamma \vdash (v_0 = v_0)$. Hence by Lemma 4.1, $\Gamma$ is inconsistent.

Alternate proof (due to a couple of students). Suppose that $\Gamma$ is consistent. By the completeness theorem let $A$ be a model of $\Gamma$. Taking $\varphi$ to be $\neg Rv_0v_0$, we get $A \models \exists v_1 \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0]$. Let $a : \omega \rightarrow A$ be any assignment. Then by Proposition 2.8(iv) there is a $b \in A$ such that $A \models \forall v_0 [Rv_0v_1 \leftrightarrow \neg Rv_0v_0][a_1^b]$. By the definition of satisfaction of $\forall$, it follows that for any $c \in A$ we have $A \models [Rv_0v_1 \leftrightarrow \neg Rv_0v_0][a_1^b]$. Hence $(c, b) \in R^A$ iff $(c, b) \notin R^A$, contradiction.

E4.3 Show that the first-order deduction theorem fails if the condition that $\varphi$ is a sentence is omitted. Hint: take $\Gamma = \emptyset$, let $\varphi$ be the formula $v_0 = v_1$, and let $\psi$ be the formula $v_0 = v_2$.

$\{v_0 = v_1\} \vdash v_0 = v_1$,

$\{v_0 = v_1\} \vdash \forall v_1 (v_0 = v_1)$,

$\{v_0 = v_1\} \vdash \forall v_1 (v_0 = v_1) \rightarrow v_0 = v_2$ by Theorem 3.27,

$\{v_0 = v_1\} \vdash v_0 = v_2$. 

On the other hand, let \( \mathcal{A} \) be the structure with universe \( \omega \) and define \( a = (0, 0, 1, 1, \ldots) \). Clearly \( \mathcal{A} \not\models [v_0 = v_1 \rightarrow v_0 = v_2][a] \). Hence \( \not\models v_0 = v_1 \rightarrow v_0 = v_2 \) by Theorem 3.2.

**E4.4** In the language for \( \mathcal{A} \) defined \( (\omega, S, 0, +, \cdot) \), let \( \tau \) be the term \( v_0 + v_1 \cdot v_2 \) and \( \nu \) the term \( v_0 + v_2 \). Let \( a \) be the sequence \( (0, 1, 2, \ldots) \). Let \( \rho \) be obtained from \( \tau \) by replacing the occurrence of \( v_1 \) by \( \nu \).

(a) Describe \( \rho \) as a sequence of integers.
(b) What is \( \rho_{\mathcal{A}}(a) \)?
(c) What is \( \nu_{\mathcal{A}}(a) \)?
(d) Describe the sequence \( a^1_{\nu_{\mathcal{A}}(a)} \) as a sequence of integers.
(e) Verify that \( \rho_{\mathcal{A}}(a) = \tau_{\mathcal{A}}(a^1_{\nu_{\mathcal{A}}(a)}) \) (cf. Lemma 4.4.)

(a) \( \rho \) is \( v_0 + (v_0 + v_2) \cdot v_2 \); as a sequence of integers it is \( \langle 7, 5, 9, 7, 5, 15, 15 \rangle \).
(b) \( \rho_{\mathcal{A}}(a) = 0 + (0 + 2) \cdot 2 = 4 \).
(c) \( \nu_{\mathcal{A}}(a) = 0 + 2 = 2 \).
(d) \( a^1_{\nu_{\mathcal{A}}(a)} = (0, 2, 2, 3, \ldots) \).
(e) \( \rho_{\mathcal{A}}(a) = 4 \), as above; \( \tau_{\mathcal{A}}(a^1_{\nu_{\mathcal{A}}(a)}) = 0 + 2 \cdot 2 = 4 \).

**E4.5** In the language for \( \mathcal{A} \) defined \( (\omega, S, 0, +, \cdot) \), let \( \varphi \) be the formula \( \forall v_0(v_0 \cdot v_1 = v_1) \), let \( \nu \) be the formula \( v_1 + v_1 \), and let \( a = (1, 0, 1, 0, \ldots) \).

(a) Describe \( \text{Subf}^1_{\nu} \varphi \) as a sequence of integers.
(b) What is \( \nu_{\mathcal{A}}(a) \)?
(c) Describe \( a^1_{\nu_{\mathcal{A}}(a)} \) as a sequence of integers.
(d) Determine whether \( \mathcal{A} \models \text{Subf}^1_{\nu} \varphi[a] \) or not.
(e) Determine whether \( \mathcal{A} \models \varphi[a^1_{\nu_{\mathcal{A}}(a)}] \) or not.

(a) \( \text{Subf}^1_{\nu} \varphi \) is \( \forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1) \); as a sequence of integers it is

\[ \langle 4, 5, 3, 9, 5, 7, 10, 10, 7, 10, 10 \rangle. \]

(b) \( \nu_{\mathcal{A}}(a) = (v_1 + v_1)_{\mathcal{A}}(\langle 1, 0, 1, 0, \ldots \rangle) = 0 + 0 = 0 \).
(c) \( a^1_{\nu_{\mathcal{A}}(a)} = (1, 0, 1, 0, \ldots) \).
(d) \( \mathcal{A} \models \text{Subf}^1_{\nu} \varphi[a] \) iff \( \mathcal{A} \models [\forall v_0(v_0 \cdot (v_1 + v_1) = v_1 + v_1)](\langle 1, 0, 1, 0, \ldots \rangle) \) iff for all \( a \in \omega \), \( a \cdot (0 + 0) = 0 + 0 \); this is true.
(e) \( \mathcal{A} \models \varphi[a^1_{\nu_{\mathcal{A}}(a)}] \) iff \( \mathcal{A} \models [\forall v_0(v_0 \cdot v_1 = v_1)](\langle 1, 0, 1, 0, \ldots \rangle) \) iff for all \( a \in \omega \), \( a \cdot 0 = 0 \); this is true.

**E4.6** Show that the condition in Lemma 4.6 that

no free occurrence of \( v_i \) in \( \varphi \) is within a subformula of the form \( \forall v_k \mu \) with \( v_k \) a variable occurring in \( \nu \)

is necessary for the conclusion of the lemma.
In the language for $\mathcal{L} = (\omega, S, 0, +, \cdot)$, let $\varphi$ be the formula $\exists v_1 [Sv_1 = v_0]$, $\nu = v_1$, and $a = 1, 1, \ldots)$. Note that the condition on $v_0$ fails. Now $\text{Subf}_{v_1} \varphi$ is the formula $\exists v_1 [Sv_1 = v_1]$, and there is no $a \in \omega$ such that $Sa = a$, and hence $\mathcal{A} \not\models \text{Subf}_{v_1} \varphi[a]$. Also, $\nu^A(a) = v_1^A(a) = a_1 = 1$, and hence $a_0^{\nu^A(a)} = 1$. Since $S0 = 1$, it follows that $\mathcal{A} \not\models \varphi[a_0^{\nu^A(a)}]$.

**E4.7** Let $\mathcal{A}$ be an $\mathcal{L}$-structure, with $\mathcal{L}$ arbitrary. Define $\Gamma = \{ \varphi : \varphi$ is a sentence and $\mathcal{A} \models \varphi[a]$ for some $a : \omega \to A \}$. Prove that $\Gamma$ is complete and consistent.

Note by Lemma 4.4 that $\mathcal{A} \models \varphi[a]$ for some $a : \omega \to A$ iff $\mathcal{A} \models \varphi[a]$ for every $a : \omega \to A$. Let $\varphi$ be any sentence. Take any $a : \omega \to A$. If $\mathcal{A} \models \varphi[a]$, then $\varphi \in \Gamma$ and hence $\Gamma \vdash \varphi$. Suppose that $\mathcal{A} \not\models \varphi[a]$. Then $\mathcal{A} \models \neg \varphi[a]$, hence $\neg \varphi \in \Gamma$, hence $\Gamma \vdash \neg \varphi$.

This shows that $\Gamma$ is complete. Suppose that $\Gamma$ is not consistent. Then $\Gamma \vdash \neg (v_0 = v_0)$ by Lemma 4.1. Then $\Gamma \models \neg (v_0 = v_0)$ by Theorem 3.2. Since $\mathcal{A}$ is a model of $\Gamma$, it is also a model of $\neg (v_0 = v_0)$, contradiction.

**E4.8** Call a set $\Gamma$ strongly complete iff for every formula $\varphi$, $\Gamma \vdash \varphi$ or $\Gamma \vdash \neg \varphi$. Prove that if $\Gamma$ is strongly complete, then $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$.

Assume that $\Gamma$ is strongly complete. Then $\Gamma \vdash v_0 = v_1$ or $\Gamma \vdash \neg (v_0 = v_1)$. If $\Gamma \vdash v_0 = v_1$, then by generalization, $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$. Suppose that $\Gamma \vdash \neg (v_0 = v_1)$. Then by generalization, $\Gamma \vdash \forall v_0 \neg (v_0 = v_1)$. By Theorem 3.27, $\Gamma \vdash \forall v_0 \neg (v_0 = v_1) \to \neg (v_1 = v_1)$. Hence $\Gamma \vdash \neg (v_1 = v_1)$. But also $\Gamma \vdash v_1 = v_1$ by Proposition 3.4, so $\Gamma$ is inconsistent by Lemma 4.1, and hence again $\Gamma \vdash \forall v_0 \forall v_1 (v_0 = v_1)$.

**E4.9** Prove that if $\Gamma$ is rich, then for every term $\sigma$ with no variables occurring in $\sigma$ there is an individual constant $c$ such that $\Gamma \vdash \sigma = c$.

By richness we have $\Gamma \vdash \exists v_0 (v_0 = \sigma) \to c = \sigma$ for some individual constant $c$. Then using (L4) it follows that $\Gamma \vdash c = \sigma$.

**E4.10** Prove that if $\Gamma$ is rich, then for every sentence $\varphi$ there is a sentence $\psi$ with no quantifiers in it such that $\Gamma \vdash \varphi \leftrightarrow \psi$.

We proceed by induction on the number $m$ of symbols $\neg, \to, \forall$ in $\varphi$. (More exactly, by the number of the integers 1,2,4 that occur in the sequence $\varphi$.) If $m = 0$, then $\varphi$ is atomic and we can take $\psi = \varphi$. Assume the result for $m$ and suppose that $\varphi$ has $m + 1$ integers 1,2,4 in it. Then there are three possibilities. First, $\varphi = \neg \varphi'$. Let $\psi'$ be a quantifier-free sentence such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$. Then $\Gamma \vdash \varphi \leftrightarrow \neg \psi'$. Second, $\varphi = (\varphi' \to \varphi'')$. Choose quantifier-free sentences $\psi'$ and $\psi''$ such that $\Gamma \vdash \varphi' \leftrightarrow \psi'$ and $\Gamma \vdash \varphi'' \leftrightarrow \psi''$. Then $\Gamma \vdash \varphi \leftrightarrow (\psi' \to \psi'')$. Third, $\varphi = \forall v_i \varphi'$. By richness, let $c$ be an individual constant such that $\Gamma \vdash \exists v_i \neg \varphi' \to \text{Subf}^{v_i}_c \neg \varphi'$. Then by Theorem 3.33 we get

1. $\Gamma \vdash \exists v_i \neg \varphi' \leftrightarrow \text{Subf}^{v_i}_c \neg \varphi'$.

Now $\text{Subf}^{v_i}_c \varphi'$ has only $m$ integers 1,2,4 in it, so by the inductive hypothesis there is a sentence $\psi$ with no quantifiers in it such that $\Gamma \vdash \text{Subf}^{v_i}_c \varphi' \leftrightarrow \psi$ and hence

2. $\Gamma \vdash \text{Subf}^{v_i}_c \neg \varphi' \leftrightarrow \neg \psi$. 

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From (1) and (2) and a tautology we get $\Gamma \vdash \neg \exists v_1 \varphi' \leftrightarrow \psi$. Then by Proposition 3.31, $\Gamma \vdash \forall v_i \varphi' \leftrightarrow \psi$, finishing the inductive proof.

**E4.11** Describe sentences in a language for ordering which say that $<$ is a linear ordering and there are infinitely many elements. Prove that the resulting set $\Gamma$ of sentences is not complete.

Let $\Gamma$ consist of the following sentences:
\[
\neg \exists v_0 (v_0 < v_0);
\forall v_0 \forall v_1 \forall v_2 [v_0 < v_1 \land v_1 < v_2 \rightarrow v_0 < v_2];
\forall v_0 \forall v_1 [v_0 < v_1 \lor v_0 = v_1 \lor v_1 < v_0];
\bigwedge_{i<j<n} \neg (v_i = v_j) \text{ for every positive integer } n.
\]

The following sentence $\varphi$ holds in $(\mathbb{Q}, <)$ but not in $(\omega, <)$:
\[
\forall v_0 \forall v_1 [v_0 < v_1 \rightarrow \exists v_2 (v_0 < v_2 \land v_2 < v_1)].
\]

Since $\varphi$ does not hold in $(\omega, <)$, we have $\Gamma \not\vdash \varphi$, by Theorem 4.2. But since $\varphi$ holds in $(\mathbb{Q}, <)$, we also have $\Gamma \not\vdash \neg \varphi$ by Theorem 4.2. So $\Gamma$ is not complete.

**E4.12** Prove that if a sentence $\varphi$ holds in every infinite model of a set $\Gamma$ of sentences, then there is an $m \in \omega$ such that it holds in every model of $\Gamma$ with at least $m$ elements.

Suppose that $\varphi$ holds in every infinite model of a set $\Gamma$ of sentences, but for every $m \in \omega$ there is a model $M$ of $\Gamma$ with at least $m$ elements such that $\varphi$ does not hold in $M$. Let $\Delta$ be the following set:
\[
\Gamma \cup \left\{ \bigwedge_{i<j<n} \neg (v_i = v_j) : n \text{ a positive integer} \right\} \cup \{\neg \varphi\}.
\]

Our hypothesis implies that every finite subset $\Delta'$ of $\Delta$ has a model; for if $m$ is the maximum of all $n$ such that the above big conjunction is in $\Delta'$, then the hypothesis yields a model of $\Delta'$. By the compactness theorem we get a model $\overline{N}$ of $\Delta$. Thus $\overline{N}$ is an infinite model of $\Gamma$ in which $\varphi$ does not hold, contradiction.

**E4.13** Let $L$ be the language of ordering. Prove that there is no set $\Gamma$ of sentences whose models are exactly the well-ordering structures.

Suppose there is such a set. Let us expand the language $L$ to a new one $L'$ by adding an infinite sequence $c_m$, $m \in \omega$, of individual constants. Then consider the following set $\Theta$ of sentences: all members of $\Gamma$, plus all sentences $c_{m+1} < c_m$ for $m \in \omega$. Clearly every finite subset of $\Theta$ has a model, so let $\overline{A} = (A, <, a_i)_{i<\omega}$ be a model of $\Theta$ itself. (Here $a_i$ is the 0-ary function, i.e., element of $A$, corresponding to $c_i$.) Then $a_0 > a_1 > \cdots$; so $\{a_i : i \in \omega\}$ is a nonempty subset of $A$ with no least element, contradiction.
Suppose that $\Gamma$ is a set of sentences, and $\varphi$ is a sentence. Prove that if $\Gamma \vdash \varphi$, then $\Delta \models \varphi$ for some finite $\Delta \subseteq \Gamma$.

We prove the contrapositive: Suppose that for every finite subset $\Delta$ of $\Gamma$, $\Delta \not\models \varphi$. Thus every finite subset of $\Gamma \cup \{\neg \varphi\}$ has a model, so $\Gamma \cup \{\neg \varphi\}$ has a model, proving that $\Gamma \not\models \varphi$.

Suppose that $f$ is a function mapping a set $M$ into a set $N$. Let $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$. Prove that $R$ is an equivalence relation on $M$.

If $a \in M$, then $f(a) = f(a)$, so $(a, a) \in R$. Thus $R$ is reflexive on $M$. Suppose that $(a, b) \in R$. Then $f(a) = f(b)$, so $f(b) = f(a)$ and hence $(b, a) \in R$. Thus $R$ is symmetric. Suppose that $(a, b) \in R$ and $(b, c) \in R$. Then $f(a) = f(b)$ and $f(b) = f(c)$, so $f(a) = f(c)$ and hence $(a, c) \in R$.

Suppose that $R$ is an equivalence relation on a set $M$. Prove that there is a function $f$ mapping $M$ into some set $N$ such that $R = \{(a, b) : a, b \in M \text{ and } f(a) = f(b)\}$.

Let $N$ be the collection of all equivalence classes under $R$. For each $a \in M$ let $f(a) = [a]_R$. Then $(a, b) \in R$ iff $a, b \in M$ and $[a]_R = [b]_R$ iff $a, b \in M$ and $f(a) = f(b)$.

Let $\Gamma$ be a set of sentences in a first-order language, and let $\Delta$ be the collection of all sentences holding in every model of $\Gamma$. Prove that $\Delta = \{\varphi : \varphi$ is a sentence and $\Gamma \vdash \varphi\}$.

For $\subseteq$, suppose that $\varphi \in \Delta$. To prove that $\Gamma \vdash \varphi$ we use the compactness theorem, proving that $\Gamma \models \varphi$. Let $\bar{A}$ be any model of $\Gamma$. Since $\varphi \in \Delta$, it follows that $\bar{A}$ is a model of $\Gamma$, as desired.

For $\supseteq$, suppose that $\varphi$ is a sentence and $\Gamma \vdash \varphi$. Then by the easy direction of the completeness theorem, $\Gamma \models \varphi$. That is, every model of $\Gamma$ is a model of $\varphi$. Hence $\varphi \in \Delta$.

Prove (2) in the proof of Theorem 4.24.

By the completeness theorem it suffices to show that

$$\models \varphi \iff \exists v_n \ldots \exists v_{n+m-1} \left[ \bigwedge_{j<m} (\sigma_j = v_{n+j}) \land Rv_n \ldots v_{n+m-1} \right].$$

So, let $\bar{A}$ be any structure, and suppose that $a : \omega \to A$. First suppose that $\bar{A} \models \varphi[a]$. Then $(\sigma_0^\bar{A}(a), \ldots, \sigma_{m-1}^\bar{A}(a)) \in R^\bar{A}$. Let 

$$b = (\cdots (a^n \sigma_0^\bar{A}(a) \sigma_1^\bar{A}(a)) \cdots)^{n+m-1}.$$

Let $j < m$. Since $n$ is greater than each $k$ such that $v_k$ occurs in $\sigma_j$, we have $\sigma_j^\bar{A}(a) = \sigma^\bar{A}(b) = b_{n+j}$. Hence $\bar{A} \models (\sigma_j = v_{n+j})[b]$, and $\bar{A} \models Rv_n \ldots v_{n+m-1}[b]$. It follows that

$$\bar{A} \models \exists v_n \ldots \exists v_{n+m-1} \left[ \bigwedge_{j<m} (\sigma_j = v_{n+j}) \land Rv_n \ldots v_{n+m-1} \right][a].$$
Thus we have shown that $\overline{A} \models \varphi[a]$ implies ($*$). Conversely, assume ($*$). Then there exist $x(0), \ldots, x(m - 1) \in A$ such that $\left[ \bigwedge_{j < m} (\sigma_j = v_{n+j}) \land R_{v_n \ldots v_{n+m-1}} \right] [b]$, where $b = (\cdots (a^n_{x(0)})^{n+1})_{x(1)} \cdots )_{x(m-1)}$. Let $j < m$. Then $\sigma_j^A(a) = \sigma_j^A(b) = b_{n+j}$. Also, we have $\langle b_n, \ldots; b_{n+m-1} \rangle \in R^\overline{A}$. So $\langle \sigma_0^A(a), \ldots, \sigma_{m-1}^A(a) \rangle \in R^\overline{A}$. Hence $\overline{A} \models \varphi[a]$. 

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