Equational theories and finite bases

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An **algebra** is a set with some basic operations on it, $\mathbf{A} = (A, f_1, f_2, \dots)$.

Ex $\quad \underline{S_3} := (S_3, \cdot, ^{-1}, 1)$

An **identity** in some fixed signature is a sentence

$$\forall x_1, \dots, x_k \ \ s(x_1, \dots, x_k) = t(x_1, \dots, x_k) \quad (s, t \text{ are terms }),$$

denoted $s \approx t$.

Ex $\quad x_1 x_2 \approx x_2 x_1$

$\mathrm{Id}_{\mathbf{A}}$:= set of identities that hold for $\mathbf{A}$
$\mathrm{Mod}(\mathrm{Id}_{\mathbf{A}}) = HSP(\mathbf{A})$ = variety generated by $\mathbf{A}$

$\mathrm{Id}_{\underline{S_3}} = \{ x^6 \approx 1, \ y^6 \approx 1, \ x_1^2 x_2^2 \approx x_2^2 x_1^2, \ \dots$

Definition

$\mathbf{A}$ is **finitely based** if there exists a finite set $\Sigma \subseteq \mathrm{Id}_{\mathbf{A}}$ such that $\Sigma \vdash \mathrm{Id}_{\mathbf{A}}$.

$x^6 \approx 1$, $x^2 y^2 \approx y^2 x^2$ + usual group axioms form a finite basis for $\mathrm{Id}_{\underline{S_3}}$.

The following finite algebras with finite signature are finitely based:

- every 2-element algebra (Lyndon 1951)
- every group (Oates, Powell 1964)
- every ring (Kruse, L'vov 1973)
- every lattice (McKenzie 1970)
- almost every algebra (Murskii 1979)

But there are nonfinitely based algebras, e.g.,

- the Brandt monoid (Perkins 1969)

$$(\{\begin{pmatrix}0&0\\0&0\end{pmatrix},\begin{pmatrix}1&0\\0&0\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\end{pmatrix},\begin{pmatrix}1&0\\0&1\end{pmatrix}\},\cdot)$$

- certain groups with an additional constant (Bryant 1982)

Question (Tarski 1960s)

Is it decidable whether a given finite algebra is finitely based?

McKenzie 1996: No

Can we describe natural classes of finitely based algebras?

Jónsson's speculation 1970s

Is every finite algebra **A** that generates a variety with finite residual bound finitely based?

Open in general; confirmed for **A** with difference term (Kearnes, Szendrei, Willard 2016).

Question

Is every finite nilpotent Mal'cev algebra **A** finitely based?

Open in general; confirmed for **A** a direct product of algebras of prime power order (Freese, McKenzie 1987).