Math 6270 - Assignment 13

Due December 4, 2019

(1) For a G-module A consider extensions

$$1 \to A \to E_i \xrightarrow{\varepsilon_i} G \to 1$$

for $i = 1, 2$. (For ease of notation assume $\mu_i \colon A \to E_i$ is just inclusion.)

The **pullback** (fiber product) of ε_1 and ε_2,

$$D := \{(x_1, x_2) \in E_1 \times E_2 \mid \varepsilon_1(x_1) = \varepsilon_2(x_2)\},$$

is an extension of $A \times A$ by G,

$$1 \to A \times A \to D \xrightarrow{\varepsilon} G \to 1$$

with $\varepsilon(x_1, x_2) := \varepsilon_1(x_1)$ for $(x_1, x_2) \in D$.

For $N := \{(a, -a) \mid a \in A\}$, its quotient

$$\bar{D} := D/N$$

is an extension of A by G,

$$1 \to A \xrightarrow{\mu} \bar{D} \xrightarrow{\bar{\varepsilon}} G \to 1$$

with $\mu(a) := (a, 0) + N$ for $a \in A$ and $\bar{\varepsilon}((x_1, x_2) + N) := \varepsilon(x_1, x_2)$ for $(x_1, x_2) \in D$. Then $\bar{D}$ is called the **Baer sum** of the extensions E_1 and E_2.

The sum of the equivalence classes of two extensions is defined as the equivalence class of their Baer sum. Argue that this gives an abelian group structure on the set of extensions of A by G modulo equivalence that is isomorphic to $H^2(G, A)$.

(2) Let D be an R-module and let

$$L \xrightarrow{\psi} M \xrightarrow{\varphi} N \to 0$$

be an exact sequence of R-modules. Show that

$$0 \to \mathrm{Hom}_R(N, D) \xrightarrow{\varphi'} \mathrm{Hom}_R(M, D) \xrightarrow{\psi'} \mathrm{Hom}_R(L, D)$$

is exact. Here $\varphi'(f) = f\varphi$ for $f \in \mathrm{Hom}_R(N, D)$, etc.

(3) Let G be an infinite cyclic group with generator g, and let A be a $\mathbb{Z}G$-module.
   (a) Prove that multiplication by $g - 1 \in \mathbb{Z}G$ yields a projective resolution of $\mathbb{Z}$ (with ε the augmentation map):

$$0 \to \mathbb{Z}G \xrightarrow{g-1} \mathbb{Z}G \xrightarrow{\varepsilon} Z \to 0$$

   (b) Show that $H^0(G, A) \cong A^G$, $H^1(G, A) \cong A/A(g-1)$ and $H^n(G, A) = 0$ for all $n \geq 2$.

(4) Read pages 830-831 in [1] on $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2)$.

References

[1] D. S. Dummit and R. M. Foote. *Abstract algebra.* John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.