Math 6270 - Assignment 4

Due September 25, 2019

(1) Read 5.1.11 amd 5.1.12 in [1] and verify that $G^{(i)} \leq \gamma_{2^i}(G)$ for all $i \in \mathbb{N}$.

(2) Let $\mathrm{GL}(n, p)$ be the multiplicative group of invertible $n \times n$-matrices over the field over prime order p. Let $\mathrm{U}(n, p)$ be the subgroup of upper unitriangular matrices, i.e., matrices with 1 in the diagonal and 0 below.
   (a) Show that $\mathrm{U}(n, p)$ is nilpotent of class $n - 1$.
   [No need to type those lengthy matrix calculations.]
   (b) Show that $\mathrm{U}(n, p)$ is a Sylow p-subgroup of $\mathrm{GL}(n, p)$.
   (c) Show that every finite p-group embeds into some $\mathrm{U}(n, p)$.

(3) Let G be a finite group with Sylow subgroup P and $N_G(P) \leq H \leq G$. Show $H = N_G(H)$.

   [Hint: Use the conjugacy part of Sylow's Theorem.]

(4) For an abelian group A the **generalized dihedral group** $\mathrm{Dih}(A)$ is the semidirect product $\mathbb{Z}_2 \ltimes_\alpha A$ where $\alpha(1)\colon A \to A, x \to x^{-1}$, acts on A by inversion.

   Show that $\mathrm{Dih}(A)$ for the Prüfer 2-group $A = \mathbb{Z}(2^\infty)$ is not nilpotent but each proper subgroup is strictly contained in its normalizer.

   [Hint: To simplify notation write the elements $\mathrm{Dih}(A)$ in the form a, ba for $a \in A$ and b acting as inversion.]

References

[1] D. J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.