Math 6270 - Assignment 3

Due September 18, 2019

(1) For p a prime, the **Prüfer p-group** $G = \mathbb{Z}(p^\infty)$ is the group of all complex p^n-th roots of unity for $n \in \mathbb{N}$ under multiplication.
   (a) Determine the list of subgroups of G.
   (b) Note that G is not finitely generated.
   (c) Show that 1 and G are the only verbal subgroups of G and that G has a fully invariant, non-verbal subgroup.

(2) Let G be a group with normal subgroups N, K.
   (a) Show that

$$h\colon G/(N \cap K) \to G/N \times G/K,\ g(N \cap K) \mapsto (gN, gK)$$

   is an injective homomorphism.
   (b) A subgroup C of a direct product $A \times B$ that projects onto each component is called a **subdirect product** of A and B.
   Show that the image of h from (a) is a subdirect product of G/N and G/K.

(3) Read [1, p. 87,88] on **characteristically simple** groups. What does this imply for minimal normal subgroups of finite solvable groups?

(4) Show that every maximal subgroup of a finite solvable group has prime power index.
   Hint: Use (3).

References

[1] D. J. S. Robinson. *A course in the theory of groups*, volume 80 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1996.