Math 6270 - Assignment 1

Due September 4, 2019

(1) Let G be a group of order $2m$ where m is odd. Show that G contains a normal subgroup of order m.
(Hint: Find an odd permutation under the regular action of G.)

(2) **Orbit counting formula.** Let G be a finite group acting on a finite set X with n orbits. For $g \in G$ let

$$\operatorname{Fix} g := \{x \in X \;:\; x \bullet g = x\}$$

be the set of fixed points of g. Show that

$$\sum_{g \in G} |\operatorname{Fix} g| = \sum_{x \in X} |G_x| = n|G|.$$

Hence the number of orbits is

$$\frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix} g|.$$

(3) Let G be a finite group, p a prime and $\operatorname{Syl}_p(G)$ the set of all Sylow p-subgroups of G.

(a) Show that

$$O_p(G) := \bigcap \operatorname{Syl}_p(G)$$

is a characteristic subgroup of G.
Recall that a subgroup H of G is **characteristic** if $\alpha(H) \leq H$ for every automorphism α of G.

(b) Show that $O_p(G)$ is the unique largest normal p-subgroup of G.

(4) Let G be a finite group with k conjugacy classes. Let $a_1, \ldots, a_k$ denote the orders of the centralizers of elements from the distinct classes.

(a) Show that

$$\frac{1}{a_1} + \cdots + \frac{1}{a_k} = 1$$

and that there exist only finitely many finite groups with k classes.

(b) Find all finite groups with 3 conjugacy classes or less.