GROUP ACTIONS AND APPLICATIONS

PETER MAYR (MATH 6270, CU BOULDER)

1. Group action

Let $(G, \cdot)$ be a group and X a non-empty set. A (right) **group action** of G on X is a map

$$X \times G \to X,\ (x, g) \mapsto x \bullet g,$$

such that

(1) $x \bullet 1 = x$ for all $x \in X$,
(2) $x \bullet (gh) = (x \bullet g) \bullet h$ for all $x \in X, g, h \in G$.

Then elements of X are often called **points**.

Let G act on X. For $g \in G$,

$$\sigma_g \colon X \to X,\ x \mapsto x \bullet g,$$

is a permutation of X, and

$$G \to \operatorname{Sym} X,\ g \mapsto \sigma_g,$$

is a group homomorphism (a **permutation representation** of G). Its kernel (the **kernel** of the action) is a normal subgroup of G. An action is **faithful** if it has trivial kernel.

Example.

(1) The **regular action** of G on itself ($X := G$) defined by right multiplication, i.e., $x \bullet g := xg$ for $x, g \in G$, is faithful.
(2) The **conjugation action** of G on itself is defined by $x \bullet g := x^g := g^{-1}xg$ for $x, g \in G$.
    Its kernel is the **center** $Z(G) := \{x \in G \mid x^g = x \text{ for all } g \in G\}$ of G.
(3) For any structure S the automorphism group $\operatorname{Aut} S$ acts faithfully on S.

Using the regular action we see

Cayley's Theorem. *Every group G embeds into the symmetric group $\operatorname{Sym} G$.*

Let G act on X. For $x \in X$

$$G_x := \{g \in G :\ x \bullet g = x\} \text{ is the } \textbf{stabilizer} \text{ of } x \text{ in } G,$$
$$O_x := \{x \bullet g :\ g \in G\} \text{ is the } \textbf{orbit} \text{ of } x \text{ in } G.$$

The G-orbits form a partition of X.

Fundamental Counting Principle. *Let G act on X and $x \in X$. Then $G_x g \mapsto x \bullet g$ is a bijection between the set of right cosets of G_x in G and the orbit of x. In particular $|O_x| = |G : G_x|$.*

Date: August 23, 2019.

2. Multiplication action

Corollary 1 (Lagrange's Theorem)**.** *Let H be a subgroup of a finite group G. Then $|H|$ divides $|G|$.*

Example. Let $H \leq G$ be a subgroup of G, and let G act on the right cosets $\{Hx : x \in G\}$ by right multiplication. Then the stabilizer of a coset Hx in G is $x^{-1}Hx =: H^x$.

The kernel of the action of G on the right cosets is the **core** of H,

$$\mathrm{core}_G(H) := \bigcap_{x \in G} H^x.$$

$\mathrm{core}_G(H)$ is the largest normal subgroup of G that is contained in H.

Theorem 2. *Let H be a subgroup of G of index $|G : H| = n$. Then H contains a normal subgroup N of G such that $|G : N|$ divides $n!$.*

3. Conjugation action

The stabilizer of $x \in G$ under conjugation is the **centralizer** $C_G(x) := \{g \in G : x^g = x\}$; its orbit is the **conjugacy class** $\{x^g : g \in G\}$ of x. Elements in the same conjugacy class are **conjugate** in G.

Corollary 3. *Let G be finite group and $x \in G$ with conjugacy class K. Then $|K| = |G : C_G(x)|$.*

For a finite group G, let k be the number of conjugacy classes of G and $n_1, \ldots, n_k$ the sizes of the conjugacy classes of G. Then

$$|G| = n_1 + \cdots + n_k$$

is called the **class equation**.

4. Sylow subgroups

Let p be a prime. A finite group whose order is a power of p is called a p-**group**.

Lemma 4. *Any non-trivial finite p-group has a non-trivial center.*

A subgroup S of a finite group G is a **Sylow p-subgroup** of G if $|S|$ is a p-group and its index $|G : S|$ is coprime to p.

Sylow's Theorem. *Let G be a finite group and p prime. Then*

1. *Every p-subgroup of G is contained in a Sylow p-subgroup of G (In particular, Sylow p-subgroups exist).*
2. *If n_p is the number of Sylow p-subgroups of G, then $n_p \equiv 1 \bmod p$.*
3. *All Sylow p-subgroups of G are conjugate in G.*

Corollary 5 (Cauchy's Theorem)**.** *If a prime p divides the order of a finite group G, then G contains an element of order p.*