Post's problem

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Post's Problem

Is there a computably enumerable set A such that $\emptyset <_T A <_T \emptyset'$?

- Many natural math problems (Hilbert's Tenth Problem, Entscheidungsproblem, word problem for semigroups, ...) can be encoded into c.e. sets but are not decidedable. If the answer to Post's Problem is no, then there is only one such undecidable problem.
- Post's approach was to construct c.e. sets with "slim" complements (c.f. simple sets for many-one reductions).

The finite injury priority method

Theorem (Friedberg 1957, Muchnik 1956)

There exist incomparable c.e. Turing degrees.

Proof.

Computably enumerate sets $A, B \subseteq \mathbb{N}$ to satisfy

- R_{2e} : $\chi_A \neq \varphi_e^B$
- R_{2e+1} : $\chi_B \neq \varphi_e^A$

Unlike in previous proofs we cannot use an $\emptyset'$-oracle for $A, B \leq \emptyset'$.

Idea to meet R_{2e}:

At stage $s+1$ choose potential witness $x \notin A_s$ (the current finite approximation of A).

- If $\varphi_{e,s}^{B_s}(x) = 0$, then set $A_{s+1} := A_s \cup \{x\}$.
- If $\varphi_{e,s}^{B_s}(x) \neq 0$ for all s, then R_{2e} is satisfied as long as $x \notin A$.

Strategy

- Try to preserve the initial segment of B_s used in the computation of $\varphi_{e,s}^{B_s}(x)$ above by **placing restraints**:
  If no $y \leq \mathrm{use}_{e,s}^{B_s}(x)$ enters B at a later stage, then $\varphi_{e,s}^{B_s}(x) = \varphi_e^B(x)$ and R_{2e} stays satisfied.
- Need to consider all requirements simultanously:
  If $i < j$ (R_i has **higher priority** than R_j), then R_i may destroy the witness x for R_j (R_j **gets injured**) and we may need to pick a new witness for R_j.
- Show inductively: Each R_i gets injured only finitely many times. When all $R_{<i}$ are satisfied, we can pick witness for R_i that won't be injured later any more.

Construction

A_s, B_s denote A, B at stage s.
Stage $s = 0$: $A_0 = B_0 := \emptyset$
$r_{i,0} := 0$ for all i (length of initial segment to protect)
$N_i := \{\sharp(x, i) \ : \ x \in \mathbb{N}\}$ (disjoint witness sets for R_i)

Stage $s + 1$ even: Choose the least e such that $r_{2e,s} = 0$ and

$$\exists x \in N_{2e} \setminus A_s \; [\varphi_{e,s}^{B_s}(x) \downarrow= 0 \text{ and } \forall i < 2e \; (r_{i,s} < x)] \qquad (\dagger)$$

- $e, x < s$ if they exist (i.e. the condition is computable).
- If no such e exists, do nothing ($A_{s+1} := A_s, B_{s+1} := B_s$, $r_{i,s+1} := r_{i,s}$) and go to stage $s + 2$.
- Else choose the least x that witnesses $(\dagger)$ for e.
  $A_{s+1} := A_s \cup \{x\}$ (R_{2e} received attention and is satisfied)
  $r_{2e,s+1} := s + 1$ (restrain B to preserve $\varphi_{e,s}^{B_s}(x) \downarrow= 0$)
  $r_{i,s+1} := 0$ for all $i > 2e$ (lower priority R_i are injured, reset)
  $r_{i,s+1} := r_{i,s}$ for all $i < 2e$ (higher priority R_i are preserved)

Stage $s + 1$ odd: Like even stage for $2e \to 2e + 1$ and $A \leftrightarrow B$.

Verification

Claim 1

If R_i receives attention at some stage $s+1$ and is not ever injured later, then A, B satisfy R_i.

Proof for $i = 2e$.

- $r_{2e,t} = s+1$ for all $t > s$ by assumption.
- No R_i for $i > 2e$ enumerates any $x \leq s+1$ into B after stage $s+1$ by construction.
- So $B \cap \{0, \ldots, s+1\} = B_s \cap \{0, \ldots, s+1\}$ and

  $$\varphi_e^B(x)\downarrow = 0 \neq 1 = \chi_A(x)$$

  for the witness x for $(\dagger)$ from stage $s+1$. $\square$

Claim 2

R_i receives attention at most finitely many times and is eventually satisfied.

Proof by induction on i.

- By induction assumption, we have a minimal v such that no R_j for $j < i$ receives attention after stage v (possibly $v = 0$).
- Then $r_{i,v} = 0$.
- If R_i receives attention at some stage $s + 1 > v$, it cannot be injured later any more and is satisfied by Claim 1.
- Suppose R_i never receives attention after v. Wlog $i = 2e$.
- No R_j for $j \neq 2e$ puts any x from N_{2e} into A. Hence $N_{2e} \cap A = N_{2e} \cap A_v$.
- After stage v, R_{2e} has highest priority but never receives attention since (†) is not satisfied for any $s > v$. In particular $\varphi_{e,s}^{B_s}(x) \neq 0$ for the least $x \in N_{2e} \setminus A_v$ with $x > v$.
- Thus $\varphi_e^B(x) \neq 0 = \chi_A(x)$ and R_{2e} holds. □

This concludes the proof of the Friedberg-Muchnik Theorem.

Note

1. The construction by Friedberg-Muchnik above is called a **finite injury** argument since every requirement is injured only finitely many times.
2. No "natural" c.e. set A with $\emptyset <_T A <_T \emptyset'$ is known.

Further results without proof

- Sacks Density Theorem: For all c.e. degrees $\mathbf{a} < \mathbf{b}$ there exists a c.e. $\mathbf{c}$ such that $\mathbf{a} < \mathbf{c} < \mathbf{b}$.
  [Proof via an infinite injury priority argument.]
- Sacks Splitting Theorem: For every c.e. degree $\mathbf{a} < \mathbf{0}'$ there exist c.e. $\mathbf{b}$, $\mathbf{c}$ such that $\mathbf{a} = \mathbf{b} \vee \mathbf{c}$.
- Lachlan, Lerman, Thomason: Every countable distributive lattice embeds into the poset of c.e. degrees.
- Lachlan, Yates: There exist c.e. degrees $\mathbf{a}$, $\mathbf{b}$ without infimum.
- Lachlan, Soare: Not every finite lattice embeds into the poset of c.e. degrees.