Productive and creative sets

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Computability Theory, March 5, 2021

So far all naturally occurring c.e. sets that are not computable (halting problem, acceptance problem) are Σ^0_1-complete with respect to many-one reductions.

Question

Is every non-computable c.e. set many-one complete for Σ^0_1?

The answer requires quite a bit of preparation.

Set-theoretical structure of computable and c.e. sets

- $\{A \subseteq \mathbb{N} \ : \ A \text{ is c.e.}\}$ forms a **bounded distributive lattice** (closed under $\cap, \cup, \emptyset, \mathbb{N}$, but not under complement $\bar{\ }$)
- $\{A \subseteq \mathbb{N} \ : \ A \text{ is computable}\}$ forms a **Boolean lattice** (closed under $\cap, \cup, \bar{\ }$)

Theorem
Every infinite c.e. $A \subseteq \mathbb{N}$ has an infinite computable subset.

Proof.

- Let A be infinite c.e. enumerated by a DTM: $a_0, a_1, \ldots$
- Then $f : \mathbb{N} \to \mathbb{N},\ x \to a_x$, is total computable with $f(\mathbb{N}) = A$ (cf. HW 6).
- Define $g(0) := f(0)$
  $$g(x+1) := f\left(\mu y\ [f(y) > g(x)]\right)$$
  $g(x+1)$ is the first value output by f that's greater than $g(x)$.
- g is total computable and strictly increasing.
- $g(\mathbb{N}) \subseteq A$ is computable since
  $$\begin{aligned} y \in g(\mathbb{N}) &\text{ iff } y \in \{g(0), \ldots, g(y)\} \\ &\text{ iff } \exists x \le y : g(x) = y. \end{aligned}$$

$\square$

Reduction property

Lemma

For all c.e. $A, B \subseteq \mathbb{N}$ there exists c.e. A', B' such that

1. $A' \subseteq A, B' \subseteq B$
2. $A' \cap B' = \emptyset$
3. $A \cup B = A' \cup B'$.

Proof.

Assume $A(x) \equiv \exists y \; R(x, y)$
$B(x) \equiv \exists y \; S(x, y)$
for R, S computable.
Define $A'(x) \equiv \exists y \; [R(x, y) \wedge \forall z \leq y \; \neg S(x, z)]$
$B'(x) \equiv \exists y \; [S(x, y) \wedge \forall z < y \; \neg R(x, z)]$

- 1.,2. are clear.
- For 3. assume $A(x) \wedge B(x)$ with smallest witnesses y_A, y_B, respectively. Then $A'(x)$ if $y_A < y_B$ and $B'(x)$ if $y_A \geq y_B$.

$\square$

Π^0_1-separation

Corollary

Let A, B be Π^0_1 and $A \cap B = \emptyset$. Then there exists a computable R such that $A \subseteq R$ and $B \cap R = \emptyset$.

Proof.

HW $\square$

Creative sets

Question

How to show a c.e. set $A \subseteq \mathbb{N}$ is not computable?

Recall: Every c.e. A is of the form $W_x := \mathrm{domain}\, \varphi_x^{(1)}$.

A is **not** computable iff $\bar{A}$ is not c.e.
iff $\forall x\ \bar{A} \neq W_x$
iff $\forall x\ [W_x \subseteq \bar{A} \Rightarrow \exists y\ y \in \bar{A} \setminus W_x]$

Definition

$B \subseteq \mathbb{N}$ is **productive** if there exists a total computable function p:

$$\forall x\ W_x \subseteq B \Rightarrow p(x) \in B \setminus W_x$$

$A \subseteq \mathbb{N}$ is **creative** if A is c.e. and $\bar{A}$ is productive.

Note

productive = p produces witnesses for not being c.e.
creative = "effectively" non-computable

Example

The complement of $K = \{x \ : \ \varphi_x(x)\downarrow\}$ is productive.
Assume $W_x \subseteq \bar{K}$.

- Then $x \notin W_x$ since otherwise $\varphi_x(x)\downarrow$ yields $x \in K$.
- Further $x \notin K$. Hence $x \in \bar{K} \setminus W_x$.

Thus K is creative.

Goal

Show Σ^0_1-complete is equivalent to creative.

Σ^0_1-complete $\Rightarrow$ creative

Theorem

Every Σ^0_1-complete set C is creative.

Proof.

Assume $K \leq_m C$ via a computable f, i.e.,

$$x \in K \text{ iff } f(x) \in C$$

It remains to show that $\bar{C}$ is productive.

- Assume $W_x \subseteq \bar{C}$ for some x. Then $f^{-1}(W_x) \subseteq \bar{K}$.
- Note there exists a computable g such that

$$\varphi_x(f(y)) = \varphi_{g(x)}(y) \;\; \forall y.$$

- Then $W_{g(x)} = f^{-1}(W_x)$.
- $W_{g(x)} \subseteq \bar{K}$ implies $g(x) \in \bar{K} \setminus W_{g(x)}$ as in the previous ex.
- Thus $fg(x) \in f(\bar{K}) \subseteq \bar{C}$ and $fg(x) \notin W_x$. $\square$

Uniform Recursion Theorem

Theorem (Kleene)

For every computable $f(x, y)$ there exists a computable $h(y)$ such that

$$\forall y, z : \ \varphi_{h(y)}(z) = \varphi_{f(h(y),y)}(z)$$

Proof.

- Define a computable d by

$$\varphi_{d(x,y)}(z) := \begin{cases} \varphi_{\varphi_x(x,y)}(z) & \text{if } \varphi_x(x, y)\downarrow, \\ \text{undefined} & \text{else.} \end{cases}$$

- Then $f(d(x, y), y) = \varphi_v(x, y)$ for some v.
- $h(y) := d(v, y)$ is the required fixed point since

$$\varphi_{d(v,y)} = \varphi_{\varphi_v(v,y)} = \varphi_{f(d(v,y),y)}.$$

$\square$

Creative $\Rightarrow \Sigma^0_1$-complete

Theorem (Myhill, 1955)

A set is creative iff it is Σ^0_1-complete.

Proof

$\Leftarrow$: already done

$\Rightarrow$: Let C be creative and p total computable such that

$$\forall x \; W_x \subseteq \bar{C} \Rightarrow p(x) \in \bar{C} \setminus W_x$$

Let A be Σ^0_1. We construct a many-one reduction ph from A to C: Consider

$$\varphi_{f(x,y)}(z) := \begin{cases} 1 & \text{if } z = p(x), y \in A, \\ \text{undefined} & \text{else.} \end{cases}$$

By the Uniform Recursion Theorem, we have computable h such that

$$\varphi_{h(y)}(z) = \varphi_{f(h(y),y)}(z) \;\; \forall y, x.$$

Then

$$W_{h(y)} = \begin{cases} \{ph(y)\} & \text{if } y \in A \\ \emptyset & \text{else.} \end{cases}$$

Claim: ph is a many-one reduction from A to C.

1. Assume $y \in A$: Then $W_{h(y)} = \{ph(y)\}$ yields $W_{h(y)} \not\subseteq \bar{C}$. Hence $ph(y) \in C$.
2. Assume $y \notin A$: Then $W_{h(y)} = \emptyset$ yields $W_{h(y)} \subseteq \bar{C}$. Hence $ph(y) \in \bar{C}$.

$\square$

All Σ^0_1-complete sets are isomorphic via computable functions

Theorem (Myhill)

For every creative set C there exists a computable bijection $f : \mathbb{N} \to \mathbb{N}$ such that

$$C = f(K)$$

Proof.

Omitted. $\square$