Math 3140 - Assignment 12

Due November 17, 2021

(1) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
 (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

(2) Read pages 220-221 in [1] on the rotational symmetries of the cube:
 (a) How many are there?
 (b) Number the faces of the cube $1, \dots, 6$. What are the cycle structures for the rotational symmetries on $1, \dots, 6$? How often does each cycle structure appear in this group?

(3) How many colorings are there of the faces of a cube in 2 colors up to rotational symmetry? (Two colorings are considered equivalent when one can be obtained from the other by rotating the cube.)
Hint: exercise (2)

The next 3 problems are for defining and constructing the alternating group A_n:

(4) Let $n \in \mathbb{N}$ and $f \in S_n$. Define the *sign* of f as

$$\operatorname{sgn}(f) := \frac{\prod_{1 \leq i < j \leq n} (f(j) - f(i))}{\prod_{1 \leq i < j \leq n} (j - i)}.$$

Show that $\operatorname{sgn} \colon S_n \to (\{-1, 1\}, \cdot)$ is a homomorphism. Why is the image of sgn in $\{-1, 1\}$?

Permutations f with $\operatorname{sgn} f = 1$ are called *even*; if $\operatorname{sgn} f = -1$, then f is called *odd*.

(5) Let $n \in \mathbb{N}$. Recall that a *transposition* is a cycle of length 2 in S_n.
 (a) Show that every cycle $(a_1\, a_2 \dots a_k)$ of length k in S_n can be written as a product of transpositions. How many transpositions are needed?
 Hint: see Problem 5c on assignment 1.
 (b) Using that every permutation is a product of disjoint cycles, conclude that every permutation is a product of transpositions.

(6) Prove:
 (a) $\operatorname{sgn}(f) = -1$ for any transposition $f \in S_n$.
 Hint: Compute $\operatorname{sgn}((1\ 2))$. Then use that every transposition $f \in S_n$ is conjugate to $(1\ 2)$ and that sgn is a homomorphism to get $\operatorname{sgn}(f)$.
 (b) $\ker \operatorname{sgn} = \{f \in S_n \ : \ f \text{ is a product of an even number of transpositions}\}$.
 This is why permutation with sign 1 are called even.

$A_n := \ker \operatorname{sgn}$ is called the *alternating group* on n letters.

(7) Show every finite abelian group is the direct product of its Sylow subgroups.

References

[1] Frederick M. Goodman. Algebra: abstract and concrete. SemiSimple Press, edition 2.6, 2015. Available from `http://www.math.uiowa.edu/~goodman`