Math 3140 - Assignment 11

Due November 10, 2021

(1) Which of the following are group actions? Check the properties. Are they transitive?
  (a) $A * x := A \cdot x$ for $A \in \mathrm{GL}(n, \mathbb{R}), x \in R^n$
  (b) G on $X := G/H$ for a subgroup H of G by $g * xH := gxH$

(2) For $(G, \cdot)$ acting on a set X and $x, y \in X$, define $x \sim y$ if $\exists g \in G\colon y = gx$. Show:
  (a) $\sim$ is an equivalence relation on X.
  (b) The orbit $Gx := \{gx \ : \ g \in G\}$ is the equivalence class of x with respect to $\sim$.

(3) Show the stabilizer $\mathrm{Stab}_G(x) := \{g \in G \ : \ gx = x\}$ is a subgroup of G.

(4) (a) How many distinct necklaces can be made with 2 red, 2 blue and 2 green beads?
  (b) How many distinct necklaces can be made with 6 beads of (at most) 3 different colors?

(5) When are two elements of S_n conjugate?
  (a) Show that for any k-cycle $(a_1, a_2, \dots, a_k) \in S_n$ and any $f \in S_n$, we have

$$f(a_1, a_2, \dots, a_k)f^{-1} = (f(a_1), f(a_2), \dots, f(a_k)).$$

  (b) For any two k-cycles $(a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_k) \in S_n$ explicitly give $f \in S_n$, such that

$$f(a_1, a_2, \dots, a_k)f^{-1} = (b_1, b_2, \dots, b_k).$$

The *cycle structure* of a permutation g is the length of the cycles in the cycle decomposition of g (counted with multiplicity). For example $g = (1\ 2\ 3)(4\ 5)(6\ 7)$ has cycle structure $3, 2, 2$.

Deduce that two permutations $g, h \in S_n$ are conjugate iff they have the same cycle structure.

(6) (a) How many different conjugacy classes are there in S_4?
  (b) For $g = (1\ 2)(3\ 4)$ determine $C_{S_4}(g)$, the centralizer of g in S_4.
  (c) How many elements in S_6 are conjugate to $(1\ 2)(3\ 4)$?