Math 3140 - Assignment 10

Due November 1, 2021

This assignment is a set of practice problems for the midterm exam on November 3.

(1) Let N be a normal subgroup of G such that G/N is abelian. Show that $x^{-1}y^{-1}xy \in N$ for all $x, y \in G$.

The expression $x^{-1}y^{-1}xy$ is called the *commutator* of x and y and denoted by $[x, y]$.

(2) Let G' be the normal subgroup of G that is generated by the set of all commutators $\{[x, y] : x, y \in G\}$. Show that G/G' is abelian.

G' is called the *commutator subgroup* or *derived subgroup* of G.

(3) By (1) and (2) the commutator subgroup G' is the smallest normal subgroup N of G such that G/N is abelian.

Use this and what you know about normal subgroups of the following groups to determine G' for

(a) G abelian,
(b) $G = S_3$,
(c) $G = D_8$.

You do not need to compute any commutators $[x, y]$ for this.

(4) An *automorphism* of a group G is an isomorphism from G to G. Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G.

(a) Determine the automorphisms of $\mathbb{Z}_8$.
(b) Show that $\mathrm{Aut}(G)$ is a group (under composition).

(5) For any $g \in G$, show that

$$c_g \colon G \to G, x \mapsto gxg^{-1},$$

is an automorphism of G. These c_g are called the *inner automorphisms* of G.

What are the inner automorphisms of $\mathbb{Z}_8$?

(6) (a) Show that $\varphi \colon G \to \mathrm{Aut}(G),\ g \mapsto c_g$, is a homomorphism.
(b) Show $\ker \varphi = Z(G)$.

In general φ is not surjective. The image $\varphi(G)$ is called the *group of inner automorphisms*, denoted $\mathrm{Inn}(G)$. Conclude that $G/Z(G) \cong \mathrm{Inn}(G)$.