Math 3140 - Assignment 7

Due October 13, 2021

(1) For groups $(G, \cdot_G)$ and $(H, \cdot_H)$ show that the *direct product*

$$G \times H := \{(g, h) \ : \ g \in G, h \in H\}$$

under the operation

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 \cdot_G g_1, h_1 \cdot_H h_2)$$

for $g_1, g_2 \in G, h_1, h_2 \in H$ is a group.

(2) Let G, H be groups and $g \in G, h \in H$. What is the order of $(g, h) \in G \times H$ in terms of the orders of g and of h?
Hint: Consider infinite and finite orders.

(3) Let $m, n \in \mathbb{N}$. Give a necessary and sufficient condition for $\mathbb{Z}_m \times \mathbb{Z}_n$ to be cyclic.
Hint: Use the previous exercise.

(4) What are the conjugacy classes in S_3, i.e., which elements are conjugate to each other?

(5) For a subgroup H of G and $a \in G$, let $Ha := \{ha \ : \ h \in H\}$ denote a *right coset* of H.
Show that H is normal in G iff $aH = Ha$ for all $a \in G$. (The latter condition says that every every left coset of H is also a right coset.)

(6) Show that every subgroup H of index 2 in a group G is normal.
Hint: Use the previous exercise.

(7) Determine the kernels and images of the following homomorphisms. Which ones are injective, surjective?
  (a) $\varphi\colon \mathbb{Z}_6 \to \mathbb{Z}_6,\ x \mapsto 4x$
  (b) $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
  for the additive group $\mathbb{R}^2$.
  (c) $h\colon \mathbb{C}^* \to \mathbb{C}^*, x \mapsto x^2$, where $\mathbb{C}^*$ denotes the multiplicative group of the complex numbers without 0.