Math 3140 - Assignment 9

Due October 26, 2016

This assignment is a set of practice problems for the midterm exam on October 26.

(1) An *automorphism* of a group G is a bijective homomorphism from G to G. Let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G.
  (a) Determine the automorphisms of $\mathbb{Z}_4$.
  (b) Show that $\mathrm{Aut}(G)$ is a group (under composition).

(2) Recall that for every $g \in G$, the map $c_g \colon G \to G, x \mapsto gxg^{-1}$ is an automorphism of G.
  (a) Show that $\varphi \colon G \to \mathrm{Aut}(G),\ g \mapsto c_g$, is a homomorphism.
  (b) Show $\ker \varphi = Z(G)$.

  In general φ is not surjective. The image $\varphi(G)$ is called the group of *inner automorphisms*, denoted $\mathrm{Inn}(G)$. Conclude that $G/Z(G) \cong \mathrm{Inn}(G)$.

(3) Show that a subgroup N of G is normal iff it is a union of conjugacy classes in G.

(4) Use Problem 7.1 and the previous problem to find all normal subgroups of S_3.
  (a) What is $Z(S_3), S_3'$?
  (b) What is S_3/S_3' isomorphic to?
  (c) How many inner automorphisms does S_3 have?

(5) (a) Is $\mathbb{Z}_8$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$?
  (b) Is $\mathbb{Z}_4 \times \mathbb{Z}_4$ isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$?

  Hint: Count the elements of fixed order.

(6) Let $(A, +)$ be an abelian group, $a_1, \dots, a_n \in A$. Show that

$$\langle a_1, \dots, a_n \rangle = \{z_1 a_1 + \cdots + z_n a_n \ : \ z_1, \dots, z_n \in \mathbb{Z}\}.$$

  Hint: There are 2 inclusions to show. In particular, you need to prove that the set on the right hand side is a subgroup.