Math 3140 - Assignment 7

Due October 12, 2016

(1) What are the conjugacy classes in S_3, i.e., which elements are conjugate?
(2) Show that for every group G conjugacy of elements is an equivalence relation.
(3) Let $n \in \mathbb{N}$ and $f \in S_n$. Define the *sign* of f as

$$\operatorname{sgn}(f) := \frac{\prod_{1 \leq i < j \leq n}(f(j) - f(i))}{\prod_{1 \leq i < j \leq n}(j - i)}.$$

Show that $\operatorname{sgn}\colon S_n \to (\{-1, 1\}, \cdot)$ is a homomorphism. Why is the image of sgn in $\{-1, 1\}$?

Permutations f with $\operatorname{sgn} f = 1$ are called *even*; if $\operatorname{sgn} f = -1$, then f is called *odd*.
(4) Recall that a *transposition* is a cycle of length 2 in S_n for any $n \in \mathbb{N}$, and that every permutation is a product of transpositions. Prove:
   (a) $\operatorname{sgn}(f) = -1$ for any transposition $f \in S_n$
   (b) $\ker \operatorname{sgn} = \{f \in S_n \,:\, f$ is a product of an even number of transpositions$\}$

$A_n := \ker \operatorname{sgn}$ is called the *alternating group* on n letters.
(5) For $n \in \mathbb{N}$ and I the $n \times n$ identity matrix show that

$$Z(\mathrm{GL}(n, \mathbb{R})) = \{aI \,:\, a \in \mathbb{R} \setminus \{0\}\}$$

Hint: Consider products of central elements with the elementary matrices E_{ij} that are 1 in position (i, j) and 0 everywhere else.