Math 3140 - Assignment 6

Due October 5, 2016

(1) For a subgroup H of $(G, \cdot)$, define a relation $\sim$ on G by
$$a \sim b \text{ if } a^{-1}b \in H.$$
Show that $\sim$ is an equivalence relation.

(2) Show that if the order of a group G is a prime p, then it is isomorphic to $(\mathbb{Z}_p, +)$.
Hint: Use Lagrange's Theorem to find all subgroups of G and deduce that G is necessarily cyclic.

(3) For a subgroup H of G and $a \in G$, let $Ha := \{ha : h \in H\}$ denote a *right coset*.
Show that H is normal in G iff $aH = Ha$ for all $a \in G$. (The latter condition says that every every left coset of H is also a right coset.)

(4) Show that every subgroup H of index 2 in a group G is normal.
Hint: Use the previous exercise.

(5) Determine the kernels and images of the following homomorphisms. Which ones are injective, surjective?
(a) $\varphi\colon \mathbb{Z}_6 \to \mathbb{Z}_6,\ x \mapsto 4x$
(b) $\psi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$
for the additive group $\mathbb{R}^2$.