Math 3140 - Basic definitions of groups

Def 1. *$(G, \cdot)$ is a* **group** *if $G \neq \emptyset$ and $\cdot$ is a binary operation on G that is*

1. *associative,*
2. *has an identity 1,*
3. *each element $g \in G$ has an inverse g^{-1} in G.*

Def 2. *H is a* **subgroup** *of a group $(G, \cdot)$ (denoted $H \leq G$) if $H \neq \emptyset$ and $\forall x, y \in H$:*

1. *$xy \in H$,*
2. *$x^{-1} \in H$.*

Def 3. *Elements x, y are* **conjugate** *in G if $\exists g \in G\colon gxg^{-1} = y$.*

Def 4. *A subgroup H of is* **normal** *in G (denoted $H \trianglelefteq G$) if*

$$\forall g \in G\colon gHg^{-1} = H.$$

Def 5. *$\varphi\colon G \to H$ is a* **homomorphism** *from $(G, \cdot)$ to $(H, *)$ if*

$$\forall x, y \in G\colon \varphi(x \cdot y) = \varphi(x) * \varphi(y)$$

$\ker \varphi := \{x \in G \ :\ \varphi(x) = 1_H\}$ *is the* **kernel** *of φ.*
$\varphi(G)$ *is the* **image** *of φ.*

Def 6. *For $H \leq G$ and $x \in G$, the* **left coset** *of x with respect to H is*

$$xH := \{xh \ :\ h \in H\}.$$

For $N \trianglelefteq G$ the **quotient group** *G/N is the set of cosets of N in G with multiplication*

$$xN \cdot yN := xyN \text{ for } x, y \in G.$$

Def 7. *The* **direct product** *of groups G, H is $G \times H$ with multiplication*

$$(g_1, h_1) \cdot (g_2, h_2) := (g_1 g_2, h_1 h_2) \text{ for } g_1, g_2 \in G, h_1, h_2 \in H.$$