Math 3130 - Assignment 12

Due April 15, 2016

(100) [1, Section 6.1] Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Show that $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$.

(101) [1, Section 6.1] Let $\mathbf{u} \in \mathbb{R}^n$. Show that

(a) $\mathbf{u} \cdot \mathbf{u} \geq 0$,

(b) $\mathbf{u} \cdot \mathbf{u} = 0$ iff $\mathbf{u} = \mathbf{0}$.

(102) [1, Section 6.1] Let $\mathbf{u} \in \mathbb{R}^n$. Is

$$V = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{u} \cdot \mathbf{x} = 0\}$$

a subspace of $\mathbb{R}^n$? Which conditions for a subspace are fulfilled by V?

(103) [1, Section 5.3 (cf. Section 5.6, Problem 5)] Consider a population of owls feeding on a population of flying squirrels in a wood. In month k, let o_k denote the number of owls and f_k the number of flying squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3 o_k + 0.4 f_k \\ f_{k+1} &= -0.4 o_k + 1.3 f_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $\mathbf{x}_k = \begin{bmatrix} o_k \\ f_k \end{bmatrix}$. Express the population change from $\mathbf{x}_k$ to $\mathbf{x}_{k+1}$ using a matrix A. Diagonalize A.

(104) Continue the previous problem: Let the starting population be $\mathbf{x}_1 = \begin{bmatrix} o_1 \\ f_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

(a) Give an explicit formula for the populations in month $k+1$.

(b) Are the populations growing or decreasing over time? By which factor?

(c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

(105) [1, cf. Section 6.1, Problems 19, 20] Are the following true or false? Why? All vectors are in $\mathbb{R}^n$.

(a) $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$.

(b) For any scalar c, $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.

(c) For a square matrix A, vectors in $\operatorname{Col} A$ are orthogonal to vectors in $\operatorname{Nul} A$.

(d) If $\mathbf{x}$ is orthogonal to every vector in $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, then $\mathbf{x}$ is also orthogonal to every vector in $\operatorname{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

(e) If $||\mathbf{u}||^2 + ||\mathbf{v}||^2 = ||\mathbf{u} + \mathbf{v}||^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

(106) [1, Section 6.1] Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$.

(a) Compute the distance between $\mathbf{u}_1$ and $\mathbf{u}_2$ as well as between $\mathbf{u}_2$ and $\mathbf{u}_3$.

(b) Compute the angle (in degrees) between $\mathbf{u}_1$ and $\mathbf{u}_2$ as well as between $\mathbf{u}_2$ and $\mathbf{u}_3$.

(107) [1, Section 6.2] Let $\mathbf{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}$, $\mathbf{u}_3 = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$.

(a) Verify that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set.
(b) Write every unit vector $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \in \mathbb{R}^3$ as linear combination $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

(108) [1, Section 6.1] Let $V = \text{Span}\{\begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \end{bmatrix}\}$ be a subspace of $\mathbb{R}^3$. Compute the orthogonal complement of V.

References

[1] David C. Lay. Linear Algebra and Its Applications. Addison-Wesley, 4th edition, 2012.