Math 2135 - Assignment 11

Due April 12, 2019

(1) Let $A, B \in F^{n\times n}$ such that AB is invertible. Show that A and B are invertible.
Hint: Use the Invertible Matrix Theorem. Show that $\operatorname{Nul} AB = \{0\}$ implies $\operatorname{Nul} B = \{0\}$.

(2) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

(3) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

(4) Give two 3×3-matrices with determinant 6. (Hint: triangular matrices.)

(5) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

(b) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c+ra & d+rb \end{bmatrix}$.

(6) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

(7) The **cross product** of vectors $a = (a_1, a_2, a_3), b = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ is defined as $a \times b := (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

(a) Show that $a \times b$ can be expressed as cofactor expansion across the first row for the determinant for the "3×3-matrix"

$$A = \begin{bmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

where e_1, e_2, e_3 are the unit vectors of $\mathbb{R}^3$.

(b) Show that $a \times b$ is orthogonal to a and to b.

(c) Bonus: Show that $|a \times b|$ is the area of the parallelogram with sides a and b.

(8) * Prove:

Theorem. $\det A$ for any $n \times n$-matrix A can be computed by a cofactor expansion across the i-th row of A, that is,

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

Recall that A_{ij} is the $(n-1) \times (n-1)$-matrix obtained from A by removing row i and column j. Hint: Use induction on i. For the induction step from i to $i+1$, flip rows i and $i+1$ (How does this change the determinant?) and use the induction assumption.