Math 2135 - Assignment 10

Due April 5, 2019

All these problems are revising material for the second midterm on April 3. So you should try to solve them before Wednesday!

Problems 2, 3a, 4b, 5a, 6b, 8 (without explanations) are comparable in length and difficulty to an actual midterm.

(1) Show the **Rank Theorem**: For $A \in F^{m\times n}$,
$$\dim \operatorname{Col} A = \dim \operatorname{Row} A$$
$\dim \operatorname{Col} A$ is also called the **rank** of A, denoted $\operatorname{rank} A$.
Hint: Use the Theorem in Problem 7 of Assignment 6 to obtain a basis of the row space $\operatorname{Row} A$.

(2) Show that $A \in F^{n\times n}$ is invertible iff $\operatorname{rank} A = n$.

(3) Let V, W be vector spaces over F with zero vectors $0_V, 0_W$, respectively. Let $f\colon V \to W$ linear. Show
  (a) $f(0_V) = 0_W$,
  (b) $\ker f$ is a subspace of V.

(4) Let $B = (\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix})$ and $C = (\begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix})$ be bases of $\mathbb{R}^2$. Compute the change of coordinate matrices (a) $[\mathrm{id}]_{B,E}$ (b) $[\mathrm{id}]_{E,C}$ and (c) $[\mathrm{id}]_{B,C}$.

(5) Let $B = (1, x, x^2)$ be a basis of P_2 and $C = (1, x, x^2, x^3)$ be a basis of P_3.
  (a) Find the matrix $[i]_{B,C}$ for the integration map $i\colon P_2 \to P_3, p \to \int_0^x p(t)\,dt$.
  (b) Use $[i]_{B,C}$ to compute $[i(p)]_C$ and $i(p)$ for the polynomial p with $[p]_B = (-2, 3, 1)$. Compare the result with what you would get by integrating p.

(6) Give bases for (a) kernel and (b) range of i from the previous example. Is i injective, surjective, bijective?

(7) Let $f\colon V \to W$ be an isomorphism between vector spaces V, W over F, and let $B = (b_1, \dots, b_n)$ be a basis of V. Show that $f(b_1), \dots, f(b_n)$ is a basis of W.

(8) True or false? Explain your answers:
  (a) For $n > 1$, $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication forms a field.
  (b) The coordinates of $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ with respect to the standard basis are $\begin{bmatrix}x\\y\end{bmatrix}$.
  (c) The dimension of the null space of a 3×5-matrix is at least 3.
  (d) Vector spaces $\mathbb{R}^3$ and $\mathbb{R}^4$ are isomorphic.