Math 2135 - Assignment 6

Due March 1, 2019

(1) Let $v_1, \ldots, v_n$ in a vector space V. Prove: If $v_n \in \operatorname{Span}(v_1, \ldots, v_{n-1})$, then

$$\operatorname{Span}(v_1, \ldots, v_n) = \operatorname{Span}(v_1, \ldots, v_{n-1}).$$

(2) (a) An 8×5-matrix A has 4 pivot columns. Find $\dim \operatorname{Nul} A$, $\dim \operatorname{Col} A$.
  (b) If B is a 3×4-matrix, what is the largest possible dimension of $\operatorname{Col} B$? What is the smallest possible dimension of $\operatorname{Nul} B$?
  (c) If the nullspace of a 4×6-matrix C has dimension 3, what is $\dim \operatorname{Col} C$?

(3) Let V be a vector space that is spanned by a finite set of vectors $v_1, \ldots, v_n$. Show that V is finite dimensional.
  Hint: How can you obtain a finite basis for V?

(4) Prove the following or give a counter example:
  (a) A basis B for a vector space V is a linear independent list of vectors in V that is as large as possible.
  (b) If $k > \dim V$, then any set of k vectors in V are linearly dependent.

(5) The **row space** $\operatorname{Row} A$ of a matrix A is the span of the rows of A.

  Two $m \times n$-matrices A, B are **row equivalent** if one can be transferred to the other by elementary row operations.

  Show that if matrices A and B are row equivalent, then $\operatorname{Row} A = \operatorname{Row} B$.
  Hint: Check for each type of elementary row operation, that it does not change the row space.

(6) Show that if B is a matrix in row echelon form, then its non-zero rows are a basis for $\operatorname{Row} B$.

(7) Problem (5) and (6) together prove the following:

  Theorem. Let A, B be row equivalent matrices and B in row echelon form. Then the non-zero rows of B form a basis for $\operatorname{Row} A$.

  (a) Use this to find a basis for the row space of

$$A = \begin{bmatrix} 0 & 2 & -3 \\ 1 & 0 & 1 \\ 2 & 2 & -1 \end{bmatrix}.$$

  (b) What can you say about the relation between $\dim \operatorname{Col} A$ and $\dim \operatorname{Row} A$ for arbitrary matrices A?

(8) Let $B = (b_1, \ldots, b_n)$ be a basis for a vector space V over a field F and consider the coordinate mapping $V \to F^n,\ x \mapsto [x]_B$.
  (a) Show that $[c \cdot x]_B = c[x]_B$ for all $x \in V, c \in F$.
  (b) Show that the coordinate mapping is onto F^n.