Math 2135 - Assignment 2

Due February 1, 2019

(1) (a) Which of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are in the nullspace of A, $\operatorname{Null} A$?

$$\mathbf{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

(b) Solve $A\mathbf{x} = \mathbf{0}$ and give the solution in parametric vector form.

(2) Show the following:

Theorem. Suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$. Then the set of all solutions of $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{p} + \operatorname{Null} A = \{\mathbf{p} + \mathbf{v} \mid \mathbf{v} \in \operatorname{Null} A\}.$$

Hint: For the proof suppose $A\mathbf{x} = \mathbf{b}$ has a solution $\mathbf{p}$ and use 2 steps:
(a) Show that if $\mathbf{v}$ is in $\operatorname{Null} A$, then $\mathbf{p} + \mathbf{v}$ is also a solution for $A\mathbf{x} = \mathbf{b}$.
(b) Show that if $\mathbf{q}$ is a solution for $A\mathbf{x} = \mathbf{b}$, then $\mathbf{q} - \mathbf{p}$ is in $\operatorname{Null} A$.

(3) Explain why the following are not fields under the usual addition and multiplication:
(a) $\mathbb{Z}$
(b) $\{x \in \mathbb{R} \mid x \geq 0\}$
(c) $\mathbb{R}^2$ with componentwise addition and multiplication, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] := \left[\begin{smallmatrix} x_1 y_1 \\ x_2 y_2 \end{smallmatrix}\right]$

(4) Let $(F, +, \cdot)$ be a field. Using only the axioms of a field, show that for each $a \in F, a \neq 0$, there exists a *unique* element $b \in F$ such that $ab = 1$.
This b is then called the *multiplicative inverse* of a and denoted by $b =: a^{-1}$.
Hint: Suppose $ab_1 = 1$ and $ab_2 = 1$. Show $b_1 = b_2$.

(5) For $n > 1$, recall that $\mathbb{Z}_n := \{[0], [1], \ldots, [n-1]\}$ is the set of residue classes modulo n. For $a, b \in \mathbb{Z}$, we have $[a] + [b] := [a + b]$ and $[a] \cdot [b] := [ab]$.
(a) Give the operation tables for $+, \cdot$ on $\mathbb{Z}_3$.
(b) What are the additive and multiplicative identity elements (if they exist)? Give additive inverses $-[a]$ and multiplicative inverses $[a]^{-1}$ (if they exist).
(c) Is $\mathbb{Z}_3$ a field?

(6) Like the previous problem for $\mathbb{Z}_4$.

(7) Use row reduction to solve the following linear system over $\mathbb{Z}_2$ (for brevity we write $0, 1$ for the classes $[0], [1]$, respectively):

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \cdot x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

(8) Let F be a field. Show that $F^{2 \times 2}$ forms a vector space with $+$ the componentwise sum of matrices and $\cdot$ the componentwise multiplication of a matrix by a scalar. In particular, determine what is the zero vector and what is the additive inverse of a matrix in $F^{2 \times 2}$.