Math 2135 - Assignment 12

Due November 22, 2019

(1) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

(2) Are the matrices A, B, C, D in assignment 11 (7), (8) and in (1) above diagonalizable? How?

(3) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
   (a) If A has n eigenvectors, then A is diagonalizable.
   (b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
   (c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
   (d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

(4) Let $A \in F^{n\times n}$ with n eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to their multiplicities). Show that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

Hint: Consider the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$.

(5) Let $A \in F^{n\times n}$ be a triangular matrix (with 0 below the diagonal).
   (a) Show that the eigenvalues of A are its diagonal elements $a_{11}, \ldots, a_{nn}$.
   (b) Is every triangular matrix diagonalizable? Consider $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$.

(6) Let A be the standard matrix for the reflection t of $\mathbb{R}^2$ on some line g throught the origin. What are the eigenvalues and eigenvectors of A? Can A be diagonalized? Hint: Consider which vectors are scaled by a reflection.

(7) Consider a population of owls feeding on a population of squirrels. In month k, let o_k denote the number of owls and s_k the number of squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3 o_k + 0.4 s_k \\ s_{k+1} &= -0.4 o_k + 1.3 s_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$. Express the population change from x_k to x_{k+1} using a matrix A. Diagonalize A.

(8) Continue the previous problem: Let the starting population be $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.
   (a) Give an explicit formula for the populations in month $k+1$.
   (b) Are the populations growing or decreasing over time? By which factor?
   (c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?