Math 2135 - Assignment 11

Due November 15, 2019

(1) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

   (a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

   (b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

   (c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c+ra & d+rb \end{bmatrix}$.

(2) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

(3) * Prove:
Theorem. $\det A$ for any $n \times n$-matrix A can be computed by a cofactor expansion across the i-th row of A, that is,

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij}.$$

Recall that A_{ij} is the $(n-1) \times (n-1)$-matrix obtained from A by removing row i and column j. Hint: Use induction on i. For the induction step from i to $i+1$, flip rows i and $i+1$ (How does this change the determinant?) and use the induction assumption.

(4) Show for 2×2-matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ that

$$\det AB = \det A \cdot \det B$$

(5) Let $A \in F^{n \times n}$. Are the following true or false? Explain why:

   (a) If two rows or columns of A are identical, then $\det A = 0$.

   (b) For $c \in F$, $\det(cA) = c \det A$.

   (c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.

   (d) A is invertible iff 0 is not an eigenvalue of A.

(6) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h \colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

(7) Give all eigenvalues and bases for eigenspaces. Do you really need the characteristic polynomials?

(a) $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

(8) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.