Math 2135 - Assignment 10

Due November 8, 2019

Problems 1-5 are a practice exam for the second midterm on November 6. So you should try to solve them before Wednesday!

(1) Let $B = (\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix})$ and $C = (\begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}-1\\4\end{bmatrix})$ be bases of $\mathbb{R}^2$. Compute the change of coordinate matrices (a) $[\mathrm{id}]_{B,E}$ (b) $[\mathrm{id}]_{E,C}$ and (c) $[\mathrm{id}]_{B,C}$.

(2) Let $B = (1, x, x^2)$ be a basis of P_2 and $C = (1, x, x^2, x^3)$ be a basis of P_3.
   (a) Find the matrix $[i]_{B,C}$ for the integration map $i\colon P_2 \to P_3, p \to \int_0^x p(t)\,dt$.
   (b) Use $[i]_{B,C}$ to compute $[i(p)]_C$ and $i(p)$ for the polynomial p with $[p]_B = (-2, 3, 1)$. Compare the result with what you would get by integrating p.

(3) Give bases for (a) kernel and (b) range of i from the previous example. Is i injective, surjective, bijective?

(4) Let $f\colon V \to W$ be an isomorphism between vector spaces V, W over F, and let $B = (b_1, \ldots, b_n)$ be a basis of V. Show that $f(b_1), \ldots, f(b_n)$ is a basis of W.

(5) True or false? Explain your answers:
   (a) For $n > 1$, $\mathbb{R}^{n\times n}$ with the usual matrix addition and multiplication forms a field.
   (b) The coordinates of $\begin{bmatrix}x\\y\end{bmatrix} \in \mathbb{R}^2$ with respect to the standard basis are $\begin{bmatrix}x\\y\end{bmatrix}$.
   (c) The dimension of the null space of a 3×5-matrix is at least 3.
   (d) Vector spaces $\mathbb{R}^3$ and $\mathbb{R}^4$ are isomorphic.

(6) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix}0 & 1 & -3\\5 & 4 & -4\\0 & -3 & -4\end{bmatrix} \qquad B = \begin{bmatrix}1 & 0 & -3 & 0\\3 & 1 & 5 & 1\\2 & 0 & 0 & 0\\7 & 1 & -2 & 5\end{bmatrix}.$$

(7) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

(8) Give two 3×3-matrices with determinant 6. (Hint: triangular matrices.)