# Math 2135 - Assignment 9

Due November 1, 2019

(1) Let $V, W$ be vector spaces over $F$, let $B = (b_1, \dots, b_n)$ be a basis of $V$, and let $f\colon V \to W$ be linear.
Show that
$$f(V) = \operatorname{Span}(f(b_1), \dots, f(b_n)).$$

(2) Let
$$f\colon \mathbb{R}^3 \to \mathbb{R}^3, x \mapsto \begin{bmatrix} 1 & -2 & 3 \\ 2 & -1 & 3 \\ 0 & 1 & -1 \end{bmatrix} \cdot x$$

(a) Determine a basis for the range and for the kernel of $f$. What are their dimensions?
Hint: Use Problem 1.

(b) Is $f$ surjective, injective, bijective?

(3) Let $P_3$ be the vector space of polynomials of degree $\leq 3$ over $\mathbb{R}$, and let $d\colon P_3 \to P_3, p \to p'$ be the derivation map.

(a) Determine the range and the kernel of $d$.

(b) Is $d$ surjective, injective, bijective?

(4) Let $e\colon P_3 \to \mathbb{R}, p \to p(1)$, be the function that evaluates a polynomial $p$ at $x = 1$.

(a) Determine the range and the kernel of $e$.

(b) Is $e$ surjective, injective, bijective?

(5) Let $A, B \in F^{n\times n}$ such that $AB$ is invertible. Show that $A$ and $B$ are invertible.
Hint: Use the Invertible Matrix Theorem. How can you get a left inverse for $B$ from the inverse $C$ of $AB$? How to get a right inverse of $A$?

(6) Show the **Rank Theorem**: For $A \in F^{m\times n}$,
$$\dim \operatorname{Col} A = \dim \operatorname{Row} A$$
$\dim \operatorname{Col} A$ is also called the **rank** of $A$, denoted $\operatorname{rank} A$.
Hint: Use the Theorem in Problem 7 of Assignment 6 to obtain a basis of the row space $\operatorname{Row} A$.

(7) Show that $A \in F^{n\times n}$ is invertible iff $\operatorname{rank} A = n$.

(8) Let $V, W$ be vector spaces over $F$ with zero vectors $0_V, 0_W$, respectively. Let $f\colon V \to W$ linear. Show

(a) $f(0_V) = 0_W$,

(b) that $\ker f$ is a subspace of $V$.