Math 2135 - Assignment 7

Due October 18, 2019

(1) (a) Let $U = \mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $e\colon U \to \mathbb{R},\ f \mapsto f(5)$, is linear.
(b) Let V be the set of real-valued functions that can be integrated over the interval $[0, 1]$. Show that
$$i\colon V \to \mathbb{R},\ f \mapsto \int_0^1 f(x)\,dx,$$
is linear.

(2) Let U, V, W be vector spaces over a field F, and let $f\colon U \to V$ and $g\colon V \to W$ be linear mappings.
Show that the composition mapping $h\colon U \to W,\ x \mapsto g(f(x))$ is linear.

(3) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.
(a) $g : \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$
(b) $h : \mathbb{R}^2 \to \mathbb{R}^2,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

(4) Give the standard matrices for the following linear transformations:
(a) $f : \mathbb{R}^2 \to \mathbb{R}^3,\ \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x \\ -x + y \end{bmatrix}$;
(b) the function g on $\mathbb{R}^2$ that scales all vectors to half their length;
(c) the projection h of vectors in $\mathbb{R}^2$ onto the x-axis;
(d) the reflection i of vectors in $\mathbb{R}^3$ on the xy-plane.

(5) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that
$$f(\begin{bmatrix} 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}, f(\begin{bmatrix} 3 \\ 2 \end{bmatrix}) = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$
(a) Use the linearity of f to find $f(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $f(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.
(b) Determine the standard matrix of f and $f(\begin{bmatrix} x \\ y \end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.

(6) Prove for a 2×2-matrix over some field F,
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$
(a) If $ad - bc \neq 0$, then
$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$
(b) If $ad - bc = 0$, then A is not invertible.
Hint: Show that the columns of A are linearly dependent in this case.

(7) If possible, invert the following matrices

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -3 \\ 4 & -6 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ -1 & 2 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 4 & 2 & 3 \end{bmatrix}$$

(8) A **diagonal matrix** A has all entries 0 except on the diagonal, that is,

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

Under which conditions is A invertible and what is A^{-1}?