Math 2135 - Assignment 5

Due October 4, 2019

Problems 1-5 are review material for the first midterm on October 2. So you should solve them before Wednesday!

(1) Explain whether the following are true or false (give counter examples if possible):
  (a) Every vector space is a subspace of itself.
  (b) Each plane in $\mathbb{R}^3$ is a subspace.
  (c) Let U be a subspace of a vector space V. Any linear combination of vectors of U is also in U.
  (d) Let $v_1, \dots, v_n$ be in a vector space V. Then $\text{Span}(v_1, \dots, v_n)$ is the smallest subspace of V containing $v_1, \dots, v_n$.

(2) Explain whether the following are true or false (give counter examples if possible):
  (a) Vectors v_1, v_2, v_3 are linearly dependent if v_2 is a linear combination of v_1, v_3.
  (b) A subset $\{v\}$ of a vector space is linearly dependent iff $v = 0$.
  (c) Two vectors in $\mathbb{R}^3$ cannot span all of $\mathbb{R}^3$.
  (d) There exist four vectors in $\mathbb{R}^3$ that are linearly independent.

(3) Consider the vector space of functions $V = \text{Span}(\cos x, 2\cos x, \cos 2x, 3\cos 2x)$ over $\mathbb{R}$. Give a basis for V.

(4) Let $v_1, \dots, v_n$ be linearly independent in a vector space V. Show that no vector in $\text{Span}(v_1, \dots, v_n)$ can be expressed by two different linear combinations.
  Hint: Use contraposition. Assume some vector u can be written as linear combination with distinct lists of coefficients $a_1, \dots, a_n$ and $b_1, \dots, b_n$. Show that $v_1, \dots, v_n$ is linearly dependent.

(5) Prove the following or give a counter example:
  (a) In general, the column space and the row space of a matrix A are not the same.
  (b) If a matrix B is a row echelon form of A, then the pivot columns of B are a basis for $\text{Col}\, A$.

(6) (a) Find vectors $u_1, \dots, u_k$ such that $\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, u_1, \dots, u_k\right)$ is a basis of $\mathbb{R}^3$.
  (b) Find vectors $v_1, \dots, v_k$ such that $\left(\begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}, v_1, \dots, v_k\right)$ is a basis of $\mathbb{R}^3$.
  Check that your choices form bases.

(7) Let $B = \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix}\right)$ be a basis of $\mathbb{R}^2$.
  (a) Find vectors $u, v \in \mathbb{R}^2$ with $[u]_B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, [v]_B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.
  (b) Compute the coordinates relative to B of $w = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

(8) Let $B = (1, x, x^2)$ and $C = (1, 1+x, 1+x+x^2)$ be bases of P_2 (polynomial functions of degree ≤ 2).

(a) Determine the polynomials p, q with $[p]_B = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$ and $[q]_C = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

(b) Compute $[r]_B$ and $[r]_C$ for $r = 3 + 2x + x^2$.