Math 2135 - Assignment 4

Due September 27, 2019

(1) Let $v_1, \dots, v_n$ be vectors in a vector space V over some field F. Complete the proof from class that $U := \text{Span}(v_1, \dots, v_n)$ is a subspace of V.
Solution:
We show the 3 conditions for being a subspace.

(a) The zero vector can be written as linear combination $\mathbf{0} = 0v_1 + \ldots + 0v_n$. Thus $\mathbf{0} \in H$.

(b) Let u and w be arbitary vectors in H. We can write these vectors as

$$u = a_1v_1 + \ldots + a_nv_n \quad \text{for some } a_1, \dots, a_n \in F,$$
$$w = b_1v_1 + \ldots + b_nv_n \quad \text{for some } b_1, \dots, b_n \in F.$$

Now

$$\begin{aligned} u + w &= a_1v_1 + \ldots + a_nv_n + b_1v_1 + \ldots + b_nv_n \\ &= (a_1 + b_1)v_1 + \ldots + (a_n + b_n)v_n. \end{aligned}$$

Thus $u + w$ is spanned by $v_1, \dots, v_n$ and hence an element of H.

(c) Let $u \in H$ as above, and let $r \in F$. Then

$$\begin{aligned} ru &= r(a_1v_1 + \ldots + a_nv_n) \\ &= ra_1v_1 + \ldots + ra_nv_n. \end{aligned}$$

Thus ru is spanned by $v_1, \dots, v_n$ and hence an element of H.

□

(2) Let $A \in F^{m \times n}$ for a field F. Prove that the nullspace of A, $\text{Nul}\, A$, is a subspace of F^n.
Hint: Use problem (2) of HW 1.
Solution:
We show the 3 conditions for being a subspace.

(a) The zero vector is clearly in $\text{Nul}(A)$ since $A\mathbf{0} = \mathbf{0}$.

(b) Let u and w be arbitary vectors in $\text{Nul}(A)$. Then $Au = \mathbf{0}$ and $Aw = \mathbf{0}$. We show that $u + w$ is in $\text{Nul}(A)$.

$$A(u + w) = Au + Aw = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

So $u + w$ is in $\text{Nul}(A)$.

(c) Let $r \in F$. Then

$$A(ru) = r(Au) = r\mathbf{0} = \mathbf{0}.$$

Hence ru is in $\text{Nul}(A)$.

□

(3) Which of the following are subspaces of the vector space $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ of all functions over $\mathbb{R}$? Check all subspace properties or give one that is not satisfied.

(a) $\{f : \mathbb{R} \to \mathbb{R} \mid f(1) = 1\}$

(b) $\{f : \mathbb{R} \to \mathbb{R} \mid f(1) = 0\}$

(c) $\{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$

Solution:

(a) No subspace since it does not contain the zero vector, i.e., the constant 0-function.
(b) Subspace since (1) contains the constant 0-function, (2) is closed under addition [for functions f, g with $f(1) = 0$ and $g(1) = 0$, also $(f + g)(1) = 0 + 0 = 0$], (3) is closed under scalar multiples [if $c \in \mathbb{R}$ and $f(1) = 0$, then also $(cf)(0) = c0 = 0$].
(c) Subspace since (1) the constant 0-function is continuous, (2) the sum of continuous functions is continuous, (3) any scalar multiple of a continuous function is continuous.

□

(4) Which of the following sets of vectors is linearly independent?

(a) $\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ -11 \\ 0 \end{bmatrix}$

Solution:
Check whether $x_1 v_1 + x_2 v_2 + x_3 v_3 = 0$ has a non-trivial solution:

(a) Row reduce

$$\begin{bmatrix} 0 & 2 & 1 \\ -1 & 1 & 0 \\ 4 & 3 & -2 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 7 & -2 \end{bmatrix} \sim \dots$$

3 pivots, no free variables, columns are linearly independent.

(b)

$$\begin{bmatrix} 1 & 2 & -1 \\ -3 & 1 & -11 \\ 2 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 7 & -14 \\ 0 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

2 pivots, 1 free variable, columns are linearly dependent.

□

(5) Are the functions $1, x, x^2$ in the vector space $\mathbb{R}^{\mathbb{R}} = \{f : \mathbb{R} \to \mathbb{R}\}$ linearly independent?

Hint: Consider a linear combination of these functions and evaluate it at some specific points $x = 0, 1, \dots$ to get several equations to solve for the coefficients.

Solution:
Solve the functional equation $a_o 1 + a_1 x + a_2 x^2 = 0$ for unknowns $a_0, a_1, a_2 \in \mathbb{R}$. We evaluate the functions on $x = 0, 1, 2$ to get:

$$a_0 = 0$$
$$a_0 + a_1 + a_2 0$$
$$a_0 + 2a_1 + 4a_2 0$$

Row reduction yields there is only the trivial solution $a_0 = a_1 = a_2 = 0$. Thus $1, x, x^2$ are linearly independent. $\square$

(6) Which of the following are bases of $\mathbb{R}^3$? Why or why not?

$$A = \left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}\right), B = \left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\-1\\4\end{bmatrix}\right), C = \left(\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}\right)$$

Solution:
A is not a basis because 2 vectors can at most span a plane but not all of $\mathbb{R}^3$.
To check whether B is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1&2&0\\2&3&-1\\0&4&4\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&-1&-1\\0&0&0\end{bmatrix}$$

Since we have a 0-row, the vectors in B do not span $\mathbb{R}^3$. Hence B is not a basis.
To check whether C is a basis we have to see whether it spans $\mathbb{R}^3$. Row reduce

$$\begin{bmatrix}1&2&0\\2&3&1\\0&4&4\end{bmatrix} \sim \begin{bmatrix}1&2&0\\0&-1&1\\0&0&8\end{bmatrix}$$

The echelon form has no 0-row. So C spans $\mathbb{R}^3$. Further we see from the echelon form that C is linearly independent. So C is a basis. $\square$

(7) Give a basis for $\operatorname{Nul} A$ and a basis for $\operatorname{Col} A$ for

$$A = \begin{bmatrix}0&2&0&3\\1&-4&-1&0\\-2&6&2&-3\end{bmatrix}$$

Solution:
Nul A is the solution set of $Ax = 0$. So we row reduce A

$$A = \begin{bmatrix}0&2&0&3\\1&-4&-1&0\\-2&6&2&-3\end{bmatrix} \sim \begin{bmatrix}1&-4&-1&0\\0&2&0&3\\0&-2&0&-3\end{bmatrix} \sim \begin{bmatrix}1&-4&-1&0\\0&2&0&3\\0&0&0&0\end{bmatrix}$$

to get the solution $x_4 = t, x_3 = s$ (both free), $x_2 = -\frac{3}{2}t, x_1 = s - 6t$. So

$$x = s\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + t\begin{bmatrix}-6\\-\frac{3}{2}\\0\\1\end{bmatrix} \text{ and } \operatorname{Nul} A \text{ has basis } \left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}-6\\-\frac{3}{2}\\0\\1\end{bmatrix}\right)$$

For a basis of the column space Col$A$ we pick the pivot columns of A, i.e., the first and second column. So Col$A$ has basis $\left(\begin{bmatrix}0\\1\\-2\end{bmatrix}, \begin{bmatrix}2\\-4\\6\end{bmatrix}\right)$. $\square$

(8) Give 2 different bases for

$$U = \text{Span}(\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix})$$

Solution:
Row reduction yields

$$\begin{bmatrix}1 & 3 & -1\\1 & -1 & 3\\2 & 0 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 3 & -1\\0 & -4 & 4\\0 & 0 & 0\end{bmatrix}.$$

So the first 2 columns of the original matrix are pivot columns and form a basis of U.

$$B_1 = (\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix})$$

Recall that the order of basis vectors is important (coordinates!). So by flipping the vectors we get a different basis

$$B_2 = (\begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\end{bmatrix})$$

Also note that any 2 columns of the matrix are linearly independent. So any 2 distinct vectors in any order out of $\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}3\\-1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\end{bmatrix}\}$ form a basis. □