Math 2135 - Assignment 2

Due September 13, 2019

(1) Let $A \in \mathbb{R}^{3\times 2}$. Explain why $Ax = b$ cannot have a solution x for all $b \in \mathbb{R}^3$.

(2) Are the following true or false? Explain your answers.
 (a) Any system of linear equations with strictly less equations than variables has infinitely many solutions.
 (b) Different sequences of elementary row reductions can transform one matrix to distinct matrices in row echelon form.
 (c) A consistent system has exactly one solution.
 (d) There exist inconsistent homogenous systems.
 (e) If a homogenous system has strictly less equations than variables, then it has infinitely many solutions.

(3) (a) Which of the vectors u, v, w are in the nullspace of A, Nul$A$?

$$\mathrm{u} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathrm{v} = \begin{bmatrix} -2 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \mathrm{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 2 & 4 \\ 2 & -4 & 1 & 0 \\ -3 & 6 & 2 & 7 \end{bmatrix}$$

 (b) Solve $A\mathrm{x} = 0$ and give the solution in parametric vector form.

(4) Show the following:

Theorem. Suppose $A\mathrm{x} = \mathrm{b}$ has a solution p. Then the set of all solutions of $A\mathrm{x} = \mathrm{b}$ is

$$p + \mathrm{Nul}A = \{p + v \mid v \in \mathrm{Nul}A\}.$$

Hint: For the proof suppose $A\mathrm{x} = \mathrm{b}$ has a solution p and use 2 steps:
 (a) Show that if v is in Nul$A$, then p + v is also a solution for $A\mathrm{x} = \mathrm{b}$.
 (b) Show that if q is a solution for $A\mathrm{x} = \mathrm{b}$, then q − p is in Nul$A$.

(5) Explain why the following are not fields under the usual addition and multiplication:
 (a) $\mathbb{Z}$ (b) $\{x \in \mathbb{R} \mid x \geq 0\}$
 (c) $\mathbb{R}^2$ with componentwise addition and multiplication, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] := \left[\begin{smallmatrix} x_1 y_1 \\ x_2 y_2 \end{smallmatrix}\right]$

(6) Let $(F, +, \cdot)$ be a field. Using only the axioms of a field, show that for each $a \in F, a \neq 0$, there exists a *unique* element $b \in F$ such that $ab = 1$.
 This b is then called the *multiplicative inverse* of a and denoted by $b =: a^{-1}$.
 Hint: Suppose $ab_1 = 1$ and $ab_2 = 1$. Show $b_1 = b_2$.

(7) For $n > 1$, recall that $\mathbb{Z}_n := \{[0], [1], \dots, [n-1]\}$ is the set of residue classes modulo n. For $a, b \in \mathbb{Z}$, we have $[a] + [b] := [a + b]$ and $[a] \cdot [b] := [ab]$.
 (a) Give the operation tables for $+, \cdot$ on $\mathbb{Z}_3$.
 (b) What are the additive and multiplicative identity elements (if they exist)? Give additive inverses $-[a]$ and multiplicative inverses $[a]^{-1}$ (if they exist).
 (c) Is $\mathbb{Z}_3$ a field?

(8) Like the previous problem for $\mathbb{Z}_4$.