Math 2135 - Assignment 13

Due December 10, 2018

(1) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h\colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

(2) Give all eigenvalues and bases for eigenspaces. Do you need the characteristic polynomials?

(a) $A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix}$ (b) $B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

(3) Give characteristic polynomial, eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

(4) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

(5) Are the matrices A, B, C, D in (2), (3), (4) diagonalizable? How?

(6) Consider a population of owls feeding on a population of squirrels. In month k, let o_k denote the number of owls and s_k the number of squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3 o_k + 0.4 s_k \\ s_{k+1} &= -0.4 o_k + 1.3 s_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$. Express the population change from x_k to x_{k+1} using a matrix A. Diagonalize A.

(7) Continue the previous problem: Let the starting population be $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.

(a) Give an explicit formula for the populations in month $k+1$.

(b) Are the populations growing or decreasing over time? By which factor?

(c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

(8) Let $A \in F^{n\times n}$. Are the following true or false? Why?

(a) If A has n distinct eigenvectors, then A is diagonalizable.

(b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.

(c) If A is invertible, then A is diagonalizable.

(d) A is diagonalizable iff A has n eigenvalues (counting multiplicities).

(e) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.