Math 2135 - Assignment 11

Due November 16, 2018

(1) Let $f\colon V \to W$ be a linear map for vectorspaces V, W over F.
Show that the **range** (image) of f,

$$f(V) := \{f(x) \;:\; x \in V\},$$

is a subspace of W.

(2) Let $f\colon V \to W$ be a linear map for vectorspaces V, W over F.
Show that the **kernel** of f,

$$\ker f := \{x \in V \;:\; f(x) = 0\},$$

is a subspace of V.

(3) Find a basis for range and kernel for each of the following linear maps:
(a) $f\colon \mathbb{R}^3 \to \mathbb{R}^2,\ x \mapsto \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \end{bmatrix} \cdot x$
(b) $d\colon P_3 \to P_2,\ p \mapsto p'$
Use range and kernel to see whether these maps are injective, surjective, bijective.

(4) Give the matrices of the linear maps in (3) with respect to bases B, C:
(a) $B = (\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}), C = (\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix})$
(b) $B = (1, x, x^2, x^3), C = (1, x, x^2)$

(5) Let U, V, W be vector spaces over a field F, and let $f\colon U \to V$ and $g\colon V \to W$ be linear mappings.
Show that the composition mapping $h\colon U \to W,\ x \mapsto g(f(x))$ is linear.

(6) Let $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $C = (\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix})$ be bases of $\mathbb{R}^2$.
(a) Find the standard matrix for $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $[u]_B \mapsto u$.
(b) Find the standard matrix for $g\colon \mathbb{R}^2 \to \mathbb{R}^2$, $u \mapsto [u]_C$.
(c) Find the standard matrix for $h\colon \mathbb{R}^2 \to \mathbb{R}^2$, $[u]_B \mapsto [u]_C$.
Hint: $h(x) = g(f(x))$.

(7) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:
(a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.
(b) Give the matrix $[t]_{B,B}$ for the reflection with respect to the coordinate system determined by B.
(c) Use the change of coordinate matrix P_B to compute the standard matrix $[t]_{E,E}$ with respect to the standard basis $E = (e_1, e_2)$.

(8) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
Hint: Recall the matrix for the rotation around the origin in $\mathbb{R}^2$.
(b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $[s]_{B,B}$ is equal to A from (a).