Math 2135 - Assignment 10

Due November 9, 2018

(1) Let $V := \mathbb{R}^+$ be the set of positive real numbers.
  (a) Is this a vector space over the field $\mathbb{R}$ under the usual operations $+$ and scalar multiples?
  (b) Show that V is a vector space over $\mathbb{R}$ with operations

$$u \oplus v := u \cdot v \text{ for addition,}$$
$$c \odot v := v^c \text{ for scalar multiples,}$$

  with $u, v \in V, c \in \mathbb{R}$. Check all properties!

(2) Prove or give a counter example:
  (a) The union $U_1 \cup U_2$ of two subspaces U_1, U_2 of a vector space V is again a subspace of V.
  (b) For every subspace U of an n-dimensional vector space V there exists $0 \leq k \leq n$ and vectors $u_1, \ldots, u_k \in V$ such that $U = \mathrm{Span}(u_1, \ldots, u_k)$.

(3) Extend the following list of vectors to a basis of the corresponding vector space if possible:
  (a) $(\begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 6 \\ 4 \end{bmatrix})$ in $\mathbb{R}^3$
  (b) $(2 + x, 3 + 2x + x^2)$ in P_2.

(4) Let $A \in F^{m \times n}$. Assume that $f \colon F^n \to F^m,\ x \mapsto A \cdot x$, is bijective. Show that A is invertible. What is f^{-1}?
  Hint: Use the Invertible Matrix Theorem.

(5) (a) Let $U = \mathbb{R}^{\mathbb{R}}$ be the set of all functions from $\mathbb{R}$ to $\mathbb{R}$. Show that $e \colon U \to \mathbb{R},\ f \mapsto f(5)$, is linear.
  (b) Let V be the set of real-valued functions that can be integrated over the interval $[0, 1]$. Show that

$$i \colon V \to \mathbb{R},\ f \mapsto \int_0^1 f(x)\, dx,$$

  is linear.

(6) Show that the following maps are not linear by giving concrete vectors for which the defining properties of linear maps are not satisfied.
  (a) $g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} xy \\ y \end{bmatrix}$
  (b) $h : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} |x| + |y| \\ 2x \end{bmatrix}$

(7) Give the standard matrices for the following linear transformations:
  (a) $f : \mathbb{R}^2 \to \mathbb{R}^3, \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 2x + y \\ x \\ -x + y \end{bmatrix}$;
  (b) the function g on $\mathbb{R}^2$ that scales all vectors to half their length;
  (c) the projection h of vectors in $\mathbb{R}^2$ onto the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(8) Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that

$$f(\begin{bmatrix}1\\2\end{bmatrix}) = \begin{bmatrix}2\\0\\-3\end{bmatrix}, f(\begin{bmatrix}3\\2\end{bmatrix}) = \begin{bmatrix}-2\\2\\1\end{bmatrix}$$

(a) Use the linearity of f to find $f(\begin{bmatrix}1\\0\end{bmatrix})$ and $f(\begin{bmatrix}0\\1\end{bmatrix})$.

(b) Determine the standard matrix of f and $f(\begin{bmatrix}x\\y\end{bmatrix})$ for arbitrary $x, y \in \mathbb{R}$.