Math 2135 - Assignment 4

Due September 28, 2018

Solve all systems of linear equations by row reduction.

(1) Is b a linear combination of the vectors a_1, a_1, a_3?

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, a_2 = \begin{bmatrix} -2 \\ 3 \\ -2 \end{bmatrix}, a_3 = \begin{bmatrix} -6 \\ 7 \\ 5 \end{bmatrix}, b = \begin{bmatrix} 11 \\ -5 \\ 9 \end{bmatrix}$$

(2) For which values of a is b in the plane spanned by v_1 and v_2?

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, v_2 = \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix}, b = \begin{bmatrix} a \\ -3 \\ -5 \end{bmatrix}$$

(3) Let A be a 2×3 matrix with 2 pivot columns.
  (a) Does $Ax = 0$ have a nontrivial solution?
  (b) Does $Ax = b$ have a solution for every $b \in \mathbb{R}^2$?
  Explain your answers.

(4) Let $A \in \mathbb{R}^{m \times n}, p \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ such that $Ap = b$. Show that the set of solutions of $Ax = b$ is

$$p + \mathrm{Nul}A = \{p + v \ : \ v \in \mathrm{Nul}A\}.$$

  Hint:
  (a) Show that if $v \in \mathrm{Nul}A$, then $p + v$ is a solution of $Ax = b$.
  (b) Show that if q is a solution for $Ax = b$, then $q - p$ is in $\mathrm{Nul}A$.

(5) Are the following true or false? Explain your answers.
  (a) The vector $3v_1$ is a linear combination of the vectors v_1, v_2.
  (b) For $v_1, v_2 \in \mathbb{R}^3$, $\mathrm{Span}\{v_1, v_2\}$ is always a plane through the origin.
  (c) For every matrix 3×2 matrix A there exists some vector b such that $Ax = b$ is not consistent.

(6) Explain why the following are not fields under the usual addition and multiplication. Give some axiom which is not satisfied:
  (a) $\mathbb{Z}$ (b) $\{x \in \mathbb{R} \ : \ x \geq 0\}$ (c) $\mathbb{R}^{2\times 2}$

(7) Let $(F, +, \cdot)$ be a field. Using only the axioms, show that for each $a \in F, a \neq 0$, there exists a unique element $b \in F$ such that $ab = 1$.
  This b is then called *the multiplicative inverse of* a and denoted by $b = a^{-1}$.
  Hint: Let $a, b, c \in F$ such that $ab = 1$ and $ac = 1$. Show that then $b = c$.

(8) For $n > 1$, recall that $\mathbb{Z}_n := \{[0], [1], \ldots, [n-1]\}$ is the set of residue classes modulo n. For $a, b \in \mathbb{Z}$, we have $[a] + [b] := [a + b]$ and $[a] \cdots [b] := [ab]$.
  Give the operation tables for $+, \cdot$ on $\mathbb{Z}_3$ and on $\mathbb{Z}_4$. Are $\mathbb{Z}_3$, $\mathbb{Z}_4$ fields?