Math 2130 - Assignment 13

Due Dec 3, 2021

(1) (a) Let W be the subspace of $\mathbb{R}^3$ with orthonormal basis $B = (\frac{1}{3}\left[\begin{smallmatrix}2\\-1\\2\end{smallmatrix}\right], \frac{1}{\sqrt{5}}\left[\begin{smallmatrix}1\\2\\0\end{smallmatrix}\right])$. Compute the coordinates $[x]_B$ for $x = \left[\begin{smallmatrix}7\\4\\4\end{smallmatrix}\right]$ in W using dot products.

(b) Give a basis for $W^\perp$.

(c) Find the closest point to $y = \left[\begin{smallmatrix}1\\2\\3\end{smallmatrix}\right]$ in W. What is the distance from y to W?

Solution:

(a) Let $[x]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. Since B is orthonormal, $c_i = x \cdot b_i$. Hence $c_1 = 6, c_2 = \frac{15}{\sqrt{5}}$.

(b) Solving $b_1 x = 0, b_2 x = 0$ yields that $W^\perp = \text{Span}\left[\begin{smallmatrix}4\\-2\\5\end{smallmatrix}\right]$.

(c) The closest point to y in W is given by the orthogonal projection $\text{proj}_W(y)$ of y on W. This can be found by the projections of y onto the basis vectors of W

$$\text{proj}_W(y) = \text{proj}_{b_1}(y) + \text{proj}_{b_2}(y) = (y \cdot b_1)b_1 + (y \cdot b_2)b_2 = 2b_1 + \sqrt{5}b_2 = \frac{1}{3}\left[\begin{smallmatrix}5\\5\\2\end{smallmatrix}\right].$$

The distance from y to W is just the length of the component of y orthogonal to W, that is $|y - \text{proj}_W(y)|$. $\square$

(2) True or false. Explain your answers.

(a) Every orthogonal set is also orthonormal.

(b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.

(c) For each x and each subspace W, the vector $x - \text{proj}_W(x)$ is orthogonal to W.

Solution:

(a) False, $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}$ are orthogonal but not orthonormal since they don't have length 1.

(b) False, every orthonormal set is orthogonal and consists of nonzero vectors. Hence it is linearly independent.

(c) True, by definition of orthogonal projection. $\square$

(3) Let W be a subset of $\mathbb{R}^n$. Show that its orthogonal complement

$$W^\perp := \{x \in \mathbb{R}^n \mid x \text{ is orthogonal to all } w \in W\}$$

is a subspace of $\mathbb{R}^n$.

Solution:

Check the 3 properties defining subspaces:

The zero vector 0 is in $W^\perp$ since $0 \cdot w = 0$ for all $w \in W$

Let $u, v \in W^\perp$. Then $u \cdot w = 0, v \cdot w = 0$ for all $w \in W$. Hence $(u + v)w = uw + vw = 0 + 0 = 0$ for all $w \in W$ and $u + v \in W^\perp$.

Let $u \in W^\perp$ and $c \in \mathbb{R}$. Then $u \cdot w = 0$ yields $(cu) \cdot w = c(\cdot w) = c0 = 0$ for all $w \in W$. Hence $cu \in W^\perp$. $\square$

(4) Let W be a subspace of $\mathbb{R}^n$. Show that

(a) $W \cap W^\perp = 0$

(b) $\dim W + \dim W^{\perp} = n$
Hint: Let $w_1, \ldots, w_k$ be a basis of W. Use that $x \in W^{\perp}$ iff x is orthogonal to $w_1, \ldots, w_k$.

Solution:
(a) If w is in W and in $W^{\perp}$, then it is orthogonal to itself. But $w \cdot w = 0$ yields that w is the zero vector.

(b) Let $A \in \mathbb{R}^{k \times n}$ with rows $w_1, \ldots, w_k$. Then $x \in R^n$ is orthogonal to $w_1, \ldots, w_k$ iff $Ax = 0$. Hence $W^{\perp}$ is the null space of A. Since W is the row space of A and dim Row A+dim Nul $A = n$, we have $\dim W + \dim W^{\perp} = n$. $\square$

(5) Use the Gram-Schmidt algorithm to find orthonormal bases for the following subspaces:

$$U = \operatorname{Span}(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}), \qquad W = \operatorname{Span}(\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix})$$

Solution:
Subspace U: Gram-Schmidt produces an orthogonal set:

$$v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \quad v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix} - \frac{5}{5} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}$$

Normalization produces an orthonormal basis

$$(\frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{105}} \begin{bmatrix} 5 \\ 4 \\ -8 \end{bmatrix}).$$

Subspace W: Gram-Schmidt produces an orthogonal set:

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \quad v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix} - \frac{-18}{9} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

We obtain $\mathbf{0}$ because the input was not linearly independent. We remove the zero vector and normalize to obtain an orthonormal basis

$$(\frac{1}{3} \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}).$$

$\square$

(6) Use the Gram-Schmidt process to transform the vectors in an orthonormal set.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solution:

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix},$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} - \frac{-2}{16} \begin{bmatrix} -2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Normalization yields an orthonormal basis

$$(\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}).$$

$\square$

(7) Find the least squares solutions of $Ax = b$.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$, $b = \left[\begin{smallmatrix} 5 \\ 1 \\ 0 \end{smallmatrix}\right]$ (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $b = \left[\begin{smallmatrix} -1 \\ 2 \\ -3 \\ 4 \end{smallmatrix}\right]$

Solution:
Solve $A^T \cdot A \cdot \hat{x} = A^T \cdot b$.

(a) $A^T \cdot A = \begin{bmatrix} 3 & 3 \\ 3 & 9 \end{bmatrix}$, $A^T \cdot b = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$ yields a unique solution $\hat{x} = \begin{bmatrix} 2/3 \\ 4/3 \end{bmatrix}$.

(b) $A^T \cdot A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$, $A^T \cdot b = \left[\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}\right]$ yields $\hat{x} = \left[\begin{smallmatrix} 1-t/2 \\ t \\ t \end{smallmatrix}\right]$ for $t \in \mathbb{R}$.

$\square$

(8) True or false for $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$. Explain your answers.

(a) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x}$ is as close as possible to b.
(b) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x} = \hat{b}$ for $\hat{b}$ the orthogonal projection of b onto $\operatorname{Col} A$.
(c) The point in $\operatorname{Col} A$ closest to b is a least squares solution of $Ax = b$.
(d) If $Ax = b$ is consistent, then every solution x is a least squares solution.

Solution:

(a) True
(b) True. For a least squares solution $\hat{x}$ we have that $A\hat{x}$ is the point in $\operatorname{Col} A$ closest to b, that is, the orthogonal projection $\hat{b}$ of b onto $\operatorname{Col} A$.
(c) False. For a least squares solution $\hat{x}$ we have that $A\hat{x}$ is the point in $\operatorname{Col} A$ closest to b.
(d) True. If $Ax = b$, then also $A^T Ax = A^T b$. Hence x is a least squares solution.

$\square$