Math 2130 - Assignment 13

Due Dec 3, 2021

(1) (a) Let W be the subspace of $\mathbb{R}^3$ with orthonormal basis $B = \left(\frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}\right)$. Compute the coordinates $[x]_B$ for $x = \begin{bmatrix} 7 \\ 4 \\ 4 \end{bmatrix}$ in W using dot products.

(b) Give a basis for $W^\perp$.

(c) Find the closest point to $y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in W. What is the distance from y to W?

(2) True or false. Explain your answers.

(a) Every orthogonal set is also orthonormal.

(b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.

(c) For each x and each subspace W, the vector $x - \text{proj}_W(x)$ is orthogonal to W.

(3) Let W be a subset of $\mathbb{R}^n$. Show that its orthogonal complement

$$W^\perp := \{x \in \mathbb{R}^n \mid x \text{ is orthogonal to all } w \in W\}$$

is a subspace of $\mathbb{R}^n$.

(4) Let W be a subspace of $\mathbb{R}^n$. Show that

(a) $W \cap W^\perp = 0$

(b) $\dim W + \dim W^\perp = n$

Hint: Let $w_1, \ldots, w_k$ be a basis of W. Use that $x \in W^\perp$ iff x is orthogonal to $w_1, \ldots, w_k$.

(5) Use the Gram-Schmidt algorithm to find orthonormal bases for the following subspaces:

$$U = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ -7 \end{bmatrix}\right), \qquad W = \text{Span}\left(\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ 4 \end{bmatrix}\right)$$

(6) Use the Gram-Schmidt process to transform the vectors in an orthonormal set.

$$x_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 1 \\ 3 \\ -3 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(7) Find the least squares solutions of $Ax = b$.

(a) $A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ -3 \\ 4 \end{bmatrix}$

(8) True or false for $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Explain your answers.

(a) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x}$ is as close as possible to b.

(b) A least squares solution of $Ax = b$ is an $\hat{x}$ such that $A\hat{x} = \hat{b}$ for $\hat{b}$ the orthogonal projection of b onto $\text{Col}\, A$.

(c) The point in $\text{Col}\, A$ closest to b is a least squares solution of $Ax = b$.

(d) If $Ax = b$ is consistent, then every solution x is a least squares solution.