Math 2130 - Assignment 12

Due Nov 19, 2021

(1) Are the matrices A, B, C, D in (3), (4), (5) of assignment 11 diagonalizable? How?

(2) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
   (a) If A has n eigenvectors, then A is diagonalizable.
   (b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
   (c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
   (d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.
   (e) Every triangular matrix is diagonalizable.

(3) Let A be the standard matrix for the reflection t of $\mathbb{R}^2$ on some line g throught the origin. What are the eigenvalues and eigenvectors of A? Can A be diagonalized? Hint: Consider what a reflection does to specific vectors.

(4) Consider a population of owls feeding on a population of squirrels. In month k, let o_k denote the number of owls and s_k the number of squirrels. Assume that the populations change every month as follows:

$$\begin{aligned} o_{k+1} &= 0.3o_k + 0.4s_k \\ s_{k+1} &= -0.4o_k + 1.3s_k \end{aligned}$$

That is, if there would be no squirrels to hunt, only 30% of the owls would survive to the next month; if there were no owls that hunted squirrels, then the squirrel population would grow by factor 1.3 every month.

Let $x_k = \begin{bmatrix} o_k \\ s_k \end{bmatrix}$. Express the population change from x_k to x_{k+1} using a matrix A. Diagonalize A.

(5) Continue the previous problem: Let the starting population be $x_1 = \begin{bmatrix} o_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} 20 \\ 100 \end{bmatrix}$.
   (a) Give an explicit formula for the populations in month $k+1$.
   (b) Are the populations growing or decreasing over time? By which factor?
   (c) What is ratio of owls to squirrels after 12 months? After 24 months? Can you explain why?

(6) (a) Give 3 vectors of length 1 in $\mathbb{R}^3$ that are orthogonal to $u = \left[\begin{smallmatrix} 1 \\ -1 \\ 2 \end{smallmatrix}\right]$.
   (b) Which of the following sets are orthogonal? Orthonormal?

$$A = \{\begin{bmatrix} 0.6 \\ 0.8 \end{bmatrix}, \begin{bmatrix} 0.8 \\ -0.6 \end{bmatrix}\}, \qquad B = \{\frac{1}{3}\left[\begin{smallmatrix} 1 \\ -2 \\ 2 \end{smallmatrix}\right], \frac{1}{\sqrt{18}}\left[\begin{smallmatrix} 4 \\ 1 \\ -1 \end{smallmatrix}\right]\}$$

(7) (a) Let W be the subspace of $\mathbb{R}^3$ with orthonormal basis $B = (\frac{1}{3}\left[\begin{smallmatrix} 2 \\ -1 \\ 2 \end{smallmatrix}\right], \frac{1}{\sqrt{5}}\left[\begin{smallmatrix} 1 \\ 2 \\ 0 \end{smallmatrix}\right])$. Compute the coordinates $[x]_B$ for $x = \left[\begin{smallmatrix} 7 \\ 4 \\ 4 \end{smallmatrix}\right]$ in W using dot products.
   (b) Give a basis for $W^\perp$.
   (c) Find the closest point to $y = \left[\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}\right]$ in W. What is the distance from y to W?

(8) True or false. Explain your answers.
   (a) Every orthogonal set is also orthonormal.
   (b) Not every orthonormal set in $\mathbb{R}^n$ is linearly independent.
   (c) For each x and each subspace W, the vector $x - \mathrm{proj}_W(x)$ is orthogonal to W.