Math 2135 - Assignment 11

Due November 12, 2021

(1) Let $A \in F^{n\times n}$. Are the following true or false? Explain why:
  (a) If two rows or columns of A are identical, then $\det A = 0$.
  (b) For $c \in F$, $\det(cA) = c \det A$.
  (c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
  (d) A is invertible iff 0 is not an eigenvalue of A.

Solution:
(a) True. If two rows or columns of A are identical, then A is not invertible and $\det A = 0$.
(b) False. $\det(cA) = c^n \det A$ since in cA every row is multiplied by c.
(c) True. Assume A is invertible. Then $\det A \cdot A^{-1} = \det A \cdot \det A^{-1}$ by a Theorem from class. Since $\det A \cdot A^{-1} = \det I = 1$, the statement follows.
(d) True. By the Invertible Matrix Theorem A is invertible iff $\operatorname{Nul} A$ is trivial. The latter means that $\operatorname{Nul}(A - 0I) = \{0\}$, i.e. 0 is not an eigenvalue of A. □

(2) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h\colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

Solution:
We have to show that $E_{h,\lambda}$ contains the 0-vector, is closed under addition and scalar multiples. Using the linearity of h we get:
$0 \in E_{h,\lambda}$ since $h(0) = 0 = \lambda 0$
If $u, v \in E_{h,\lambda}$, then $h(u + v) = h(u) + h(v) = \lambda u + \lambda v = \lambda(u + v)$ and $u + v \in E_{h,\lambda}$.
If $v \in E_{h,\lambda}$ and $c \in F$, then $h(cv) = ch(v) = c\lambda v = \lambda cv$ and $cv \in E_{h,\lambda}$. □

(3) Give all eigenvalues and bases for eigenspaces of the following matrices. Do you need the characteristic polynomials?

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

Solution:
Since A, B are triangular matrices, their eigenvalues are just their diagonal elements.
  (a) A has eigenvalue -3 with multiplicity 2: $\operatorname{Nul}(A - (-3)I)$ has basis $(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$
  (b) B has eigenvalues $2, 0, 3$:
    $\operatorname{Nul}(A - 2I)$ has basis $(\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix})$.
    $\operatorname{Nul}(A - 0I)$ has basis $(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix})$.
    $\operatorname{Nul}(A - 3I)$ has basis $(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix})$.

□

(4) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

Solution:

Characteristic polynomial:

$$\begin{aligned} \det(C - \lambda I) &= (1-\lambda)(1-\lambda) - 2 \cdot 3 \\ &= \lambda^2 - 2\lambda - 5 \end{aligned}$$

Eigenvalues are the roots of the characteristic polynomial. Use the quadratic formula

$$\begin{aligned} \lambda_{1,2} &= 1 \pm \sqrt{1+5} \\ &= 1 \pm \sqrt{6} \end{aligned}$$

Eigenvector for $\lambda = 1 + \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} -\sqrt{6} & 2 \\ 3 & -\sqrt{6} \end{bmatrix} \sim \begin{bmatrix} -\sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and added to row 2.

So the eigenspace for $\lambda = 1 + \sqrt{6}$ has basis $(\begin{bmatrix} 2 \\ \sqrt{6} \end{bmatrix})$.

Eigenvector for $\lambda = 1 - \sqrt{6}$:

$$C - \lambda I = \begin{bmatrix} \sqrt{6} & 2 \\ 3 & \sqrt{6} \end{bmatrix} \sim \begin{bmatrix} \sqrt{6} & 2 \\ 0 & 0 \end{bmatrix}$$

where we multiplied row 1 by $\frac{3}{\sqrt{6}}$ and subtracted from row 2.

So the eigenspace for $\lambda = 1 - \sqrt{6}$ has basis $(\begin{bmatrix} -2 \\ \sqrt{6} \end{bmatrix})$.

□

(5) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

Solution:

Characteristic polynomial:

$$\begin{aligned} \det(D - \lambda I) &= (-3-\lambda) \cdot \det \begin{bmatrix} -1-\lambda & 4 \\ 6 & 9-\lambda \end{bmatrix} \\ &(-3-\lambda)[(-1-\lambda)(9-\lambda) - 24] \\ &(-3-\lambda)[\lambda^2 - 8\lambda - 33] \end{aligned}$$

Eigenvalues are $\lambda_1 = -3$ and the roots of $\lambda^2 - 8\lambda - 33$. The quadratic formula yields

$$\lambda_{2,3} = 4 \pm \sqrt{4^2 + 33}$$

So $\lambda_2 = -3$ and $\lambda_3 = 11$.

The eigenspace for $\lambda = -3$ has basis $(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix})$.

The eigenspace for $\lambda = 11$ has basis $\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right)$. □

(6) Are the matrices A, B, C, D in (3), (4), (5) diagonalizable? How?

(7) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
   (a) If A has n eigenvectors, then A is diagonalizable.
   (b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
   (c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
   (d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

(8) Let $A \in \mathbb{R}^{n \times n}$ with n eigenvalues $\lambda_1, \ldots, \lambda_n$ (repeated according to their multiplicities). Show that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

Hint: Consider the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$.

Solution:

Since $\lambda_1, \ldots, \lambda_n$ are the roots of the characteristic polynomial, the characteristic polynomial can be factored as

$$\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda).$$

Note that the signs are correct because on both sides of the equation the coefficient of λ^n is $(-1)^n$.

By plugging in $\lambda = 0$ we get

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n.$$

□