Math 2135 - Assignment 11

Due November 12, 2021

(1) Let $A \in F^{n\times n}$. Are the following true or false? Explain why:
  (a) If two rows or columns of A are identical, then $\det A = 0$.
  (b) For $c \in F$, $\det(cA) = c \det A$.
  (c) If A is invertible, then $\det A^{-1} = \frac{1}{\det A}$.
  (d) A is invertible iff 0 is not an eigenvalue of A.

(2) Eigenvalues, -vectors and -spaces can be be defined for linear maps just as for matrices.

Let $h\colon V \to W$ be a linear map for vector spaces V, W over F. Show that the eigenspace for $\lambda \in F$,

$$E_{h,\lambda} := \{x \in V \ : \ h(x) = \lambda x\},$$

is a subspace of V.

(3) Give all eigenvalues and bases for eigenspaces of the following matrices. Do you need the characteristic polynomials?

$$A = \begin{bmatrix} -3 & 1 \\ 0 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 0 & 3 \end{bmatrix}$$

(4) Give the characteristic polynomial, all eigenvalues and bases for eigenspaces for $C = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$.

(5) Compute eigenvalues and eigenvectors for $D = \begin{bmatrix} -1 & 4 & 1 \\ 6 & 9 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

(6) Are the matrices A, B, C, D in (3), (4), (5) diagonalizable? How?

(7) Let A be an $n \times n$-matrix. Are the following true or false? Explain why:
  (a) If A has n eigenvectors, then A is diagonalizable.
  (b) If a 4×4-matrix A has two eigenvalues with eigenspaces of dimension 3 and 1, respectively, then A is diagonalizable.
  (c) A is diagonalizable iff A has n eigenvalues (counting multiplicities).
  (d) If $\mathbb{R}^n$ has a basis of eigenvectors of A, then A is diagonalizable.

(8) Let $A \in \mathbb{R}^{n\times n}$ with n eigenvalues $\lambda_1, \dots, \lambda_n$ (repeated according to their multiplicities). Show that

$$\det A = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$$

Hint: Consider the characteristic polynomial $\det(A - \lambda I) = (\lambda_1 - \lambda) \cdots (\lambda_n - \lambda)$.