Math 2130 - Assignment 10

Due November 5, 2021

Problems 1-7 are review material for the second midterm on November 3. Solve them before Wednesday!

(1) Let $T\colon P_2 \to \mathbb{R}, p \mapsto p(3)$, be the map that evaluates a polynomial p at $x = 3$.
  (a) Show that T is linear.
  (b) Determine the kernel of T, that is, $\{p \in P_2 \ : \ T(p) = 0\}$, and the image of T, that is, $T(P_2)$.
  (c) Is T injective, surjective, bijective?

Solution:

(a) For linearity, let $p, q \in P_2$. Their sum $p + q$ is the polynomial that maps t to $p(t) + q(t)$. So

$$T(p + q) = (p + q)(3) = p(3) + q(3) = T(p) + T(q).$$

Further let $c \in \mathbb{R}$. Then cp maps t to $cp(t)$. So

$$T(cp) = (cp)(3) = cp(3) = cT(p).$$

Hence T is linear.

(b) The kernel of T, $\ker T$, consists of all the polynomials that evaluate to 0 at 3, that is,

$$\ker T = \{(t - 3)q : q \in P_1\}.$$

The range of T, $T(P_2)$, is $\mathbb{R}$. For every $b \in \mathbb{R}$, there exists a polynomial $p \in P_2$ that is mapped to b. Choose for example the constant polynomial $p(t) = b$.

(c) Since the kernel of T is non-trivial, T is not injective.
Since the range of T is equal to its codomain, T is surjective.
T is not bijective since it is not injective.

□

(2) Let $B = (b_1, b_2)$ with $b_1 = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}, b_2 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$ and $C = (\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix})$ be bases of a subspace H of $\mathbb{R}^3$.
  (a) Compute the coordinates $[b_1]_C$ and $[b_2]_C$.
  (b) What is the change of coordinate matrix $P_{C \leftarrow B}$?
  (c) What is the change of coordinate matrix $P_{B \leftarrow C}$?

Solution:
(a) Solve the linear system

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 11 \\ 5 \end{bmatrix}$$

to obtain $x_1 = 3, x_2 = -4$. So $[b_1]_C = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

Similarly we get $[b_2]_C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(b) Since the columns of $P_{C \leftarrow B}$ are just the coordinate tuples $[b_i]_C$, we see

$$P_{C \leftarrow B} = \begin{bmatrix} 3 & 1 \\ -4 & 1 \end{bmatrix}$$

(c)

$$P_{B \leftarrow C} = P_{C \leftarrow B}^{-1} = \frac{1}{7} \begin{bmatrix} 1 & -1 \\ 4 & 3 \end{bmatrix}$$

Alternatively we could also get $P_{B \leftarrow C}$ from its columns $[c_i]_B$. □

(3) Let $C = (1+t, t+t^2, 1+t^2)$ be a basis for P_2. Compute the coordinates $[p]_C$ for $p = 2 + t^2$.

Solution:
Solve

$$c_1(1+t) + c_2(t+t^2) + c_3(1+t^2) = 2 + t^2.$$

Comparing the coefficients on both sides of this equation yields

$$\begin{aligned} c_1 + c_3 &= 2 && \text{(constant part)} \\ c_1 + c_2 &= 0 && \text{(multiples of } t\text{)} \\ c_2 + c_3 &= 1 && \text{(multiples of } t^2\text{)} \end{aligned}$$

Solving that system of linear equations yields $c_1 = \frac{1}{2}, c_2 = -\frac{1}{2}, c_3 = \frac{3}{2}$. So $[u]_B = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$. □

(4) (a) If A is a 3×4-matrix, what is the largest possible rank of A? What is the smallest possible dimension of Nul A?

Solution:
The rank of a matrix is the number of its pivot elements, which is at most the number of its rows and at most the number of its columns. So rank$A \leq \max(3, 4) = 3$. Since the largest possible rank is 3, the smallest number of free variables in $Ax = 0$ is 1. So the dimension of Nul A is 1 or larger. □

(b) If the nullspace of a 4×6-matrix B has dimension 3, what is the dimension of the row space of B?

Solution:
dim Nul $A +$ dim Row $A =$ the number of columns of A
So dim Row $A = 3$. □

(c) Give two 3×3-matrices with determinant 6.

Solution:
Any triangular or diagonal matrix whose diagonal elements multiply to 6 will do, e.g.,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

□

(5) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} u & v \\ w & x \end{bmatrix}$. Show

$$\det(AB) = \det(A)\det(B).$$

Solution:

$$AB = \begin{bmatrix} au + bw & av + bx \\ cu + dw & cv + dx \end{bmatrix}$$

$$\det AB = (au + bw)(cv + dx) - (av + bx)(cu + dw) = \cdots = \det A \cdot \det B$$

□

(6) Let $A \in \mathbb{R}^{n \times n}$ and $\lambda \in \mathbb{R}$. Is

$$H = \{\mathbf{x} \in \mathbb{R}^n \ : \ A\mathbf{x} = \lambda\mathbf{x}\}$$

a subspace of $\mathbb{R}^n$? Which conditions for a subspace are fulfilled by H?

Solution:
Yes. We show the subspace conditions:
(1) Since $A\mathbf{0} = \mathbf{0} = \lambda\mathbf{0}$, the zero vector is in H.
(2) Let $u, v \in H$. Then $A(u+v) = Au + Av = \lambda u + \lambda v = \lambda(u+v)$. Thus $u + v \in H$.
(3) Let $r \in \mathbb{R}$. Then $A(ru) = rAu = r\lambda u = \lambda(ru)$. Thus $ru \in H$. □

(7) For which $\mu \in \mathbb{R}$ has the matrix

$$B = \begin{bmatrix} 6-\mu & 2 \\ -6 & -1-\mu \end{bmatrix}$$

a determinant $\det B = 0$?

Solution:

$$\det B = (6-\mu)(-1-\mu) - (-12) = \mu^2 + \mu - 6\mu - 6 + 12 = \mu^2 - 5\mu + 6.$$

Now $\det B = 0$ yields a quadratic equation $\mu^2 - 5\mu + 6 = 0$ whose solution is

$$\mu = \frac{5 \pm \sqrt{5^2 - 4 \cdot 6}}{2} = \frac{5}{2} \pm \frac{1}{2} \in \{2, 3\}.$$

Thus $\det B = 0$ iff $\mu \in \{2, 3\}$. □

(8) Let

$$A = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}.$$

(a) Compute the matrices $A - 2I$, $A - 3I$, and $A - I$.
(b) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = 2\mathbf{x}$. Give the parametrized vector form for the solution set.
Hint: $A\mathbf{x} = 2\mathbf{x}$ iff $A\mathbf{x} = 2I\mathbf{x}$ iff $(A - 2I)\mathbf{x} = \mathbf{0}$.
(c) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = 3\mathbf{x}$. Give the parametrized vector form.
(d) Find all $\mathbf{x} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{x}$. Give the parametrized vector form.

Solution:

(a)

$$A - 2I = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}, \quad A - 3I = \begin{bmatrix} 3 & 2 \\ -6 & -4 \end{bmatrix}, \quad A - I = \begin{bmatrix} 5 & 2 \\ -6 & -2 \end{bmatrix}.$$

(b) We solve $(A - 2I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = r \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ for $r \in \mathbb{R}$.

(c) We solve $(A - 3I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = r \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$ for $r \in \mathbb{R}$.

(d) We solve $(A - I)\mathbf{x} = \mathbf{0}$ and obtain $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

□