Math 2130 - Assignment 9

Due October 29, 2021

(1) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
 (a) Find the matrix $d_{B\leftarrow B}$ for the derivation map $d\colon P_3 \to P_3, p \to p'$.
 (b) Use $d_{B\leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

Solution:
Compute coordinates of the derivatives $d(b_i)$ for the basis vectors in B to get

$$d_{B\leftarrow B} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then $[p']_B = d_{B\leftarrow B}[p]_b = (2, 0, 3)$ and $p' = 2 + 3x^2$. □

(2) Let $B = (\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix})$ and $C = (\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.
 (a) Find the change of coordinates matrix $P_{E\leftarrow B}$ for $f: [u]_B \mapsto [u]_E$.
 (b) Find the change of coordinates matrix $P_{C\leftarrow E}$ for $g: [u]_E \mapsto [u]_C$.
 (c) Find the change of coordinates matrix $P_{C\leftarrow B}$ for $h: [u]_B \mapsto [u]_C$.
 Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$.

Solution:
Let E be the standard basis of $\mathbb{R}^2$.
 (a) How to compute E-coordinates from B-coordinates? The standard matrix for f is

$$P_{B\leftarrow E} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

 Note that the columns are exactly the vectors of B. Changing coordinates from any B to the standard basis E is easy.
 (b) How to compute C-coordinates from E-coordinates? The standard matrix for g is

$$P_{E\leftarrow C} = P_{C\leftarrow E}^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

 Note that the matrix is the inverse of the matrix whose columns are the vectors of C. For changing coordinates from the standard basis E to a basis C you need to solve a linear system or find the inverse.
 (c) How to compute C-coordinates from B-coordinates? First go from B-coordinates to E-coordinates and then to C-coordinates. The matrix for $h = g \circ f$ is

$$P_{C\Leftarrow B} = P_{C\leftarrow E}P_{E\Leftarrow B} = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ -3 & -7 \end{bmatrix}$$

□

(3) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:
(a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.
(b) Give the matrix $t_{B\leftarrow B}$ for the reflection with respect to the coordinate system determined by B.
(c) Use the change of coordinate matrix to compute the standard matrix $t_{E\leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

Solution:

(a) Pick $B = (\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix})$ with the first vector b_1 on the line $3x + y = 0$, the second b_2 orthogonal. Then $t(b_1) = b_1, t(b_2) = -b_2$.

(b)

$$t_{B\leftarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

looks like the standard matrix for the reflection on the x-axis.

(c) To get $t_{E\leftarrow E}$ from $[t]_{B,B}$ we need to multiply with change of coordinate matrices,

$$t_{E\leftarrow E} = P_{B\leftarrow E} t_{B\leftarrow B} P_{E\leftarrow B} = \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -3 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -3 \\ -3 & -1 \end{bmatrix} \frac{1}{10} \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} -8 & -6 \\ -6 & 8 \end{bmatrix}$$

□

(4) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?
(b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $s_{B\leftarrow B}$ is equal to A from (a).
(c) Give the standard matrix $s_{E\leftarrow E}$ for the standard basis E (You do not need to actually multiply and invert the involved matrices; the product formula is enough).

Solution:

(a) e_3 remains fixed, e_1, e_2 rotate like in $\mathbb{R}^2$, i.e.,

$$A = \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 & 0 \\ \sin\pi/3 & \cos\pi/3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We want $b_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (fixed under rotation) and b_1, b_2 in a plane orthogonal to b_3, orthogonal to each other and of length 1, e.g.,

$$b_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, b_2 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix} \text{ (the normalized vector for } b_3 \times b_1).$$

Note that the basis b_1, b_2, b_3 is right-handed since $b_1 \times b_2$ points in direction of b_3. For $B = (b_1, b_2, b_3)$ the matrix $s_{B\leftarrow B}$ is equal to A from (a).

(c) To get the standard matrix $s_{E\leftarrow E}$ from $s_{B\leftarrow B}$ we need to multiply with change of coordinate matrices: let

$$P_{E\leftarrow B} = [b_1, b_2, b_3] =: P$$

be the matrix with vectors b_1, b_2, b_3 in its columns. Then

$$s_{E\leftarrow E} = P_{E\leftarrow B} s_{B,B} P_{E\leftarrow B} = P \cdot A \cdot P^{-1}$$

□

(5) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ 5 & 4 & -4 \\ 0 & -3 & -4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 3 & 1 & 5 & 1 \\ 2 & 0 & 0 & 0 \\ 7 & 1 & -2 & 5 \end{bmatrix}.$$

Solution:
Expand $\det A$ down the first column:

$$\det A = 0 \cdot \det A_{11} - 5 \cdot \det A_{21} + 0 \cdot \det A_{31} = -5 \cdot \det \begin{bmatrix} 1 & -3 \\ -3 & -4 \end{bmatrix} = -5(1(-4) - (-3)(-3)) = 65$$

Expand $\det B$ across 3rd row:

$$\det B = 2 \cdot \det B_{13} = 2 \cdot \det \begin{bmatrix} 0 & -3 & 0 \\ 1 & 5 & 1 \\ 1 & -2 & 5 \end{bmatrix}$$

Expand across 1st row:

$$\det B_{13} = -1(-3) \det \begin{bmatrix} 1 & 1 \\ 1 & 5 \end{bmatrix} = 3 \cdot (1 \cdot 5 - 1 \cdot 1) = 12$$

So $\det B = 2 \cdot 12 = 24$. □

(6) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand $\det A$ across the first row.

Solution:

$$\begin{aligned}\det A &= a_{11} \cdot \det A_{11} - a_{12} \cdot \det A_{12} + a_{13} \cdot \det A_{13} \\ &= a_{11} \cdot \det \begin{bmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{bmatrix} - a_{12} \cdot \det \begin{bmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{bmatrix} + a_{13} \cdot \det \begin{bmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\ &= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}\end{aligned}$$

□

(7) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

Solution:
Interchanging 2 rows changes the sign of the determinant:

$$\det \begin{bmatrix} c & d \\ a & b \end{bmatrix} = cb - ad = -\det A$$

□

(b) How does multiplying one row by a scalar effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare $\det A$ and $\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix}$.

Solution:
Adding a multiple of the first row to another does not change the determinant:

$$\det \begin{bmatrix} a & b \\ c + ra & d + rb \end{bmatrix} = a(d + rb) - b(c + ra) = ad - bc = \det A$$

□

(8) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$

Solution:

$$\det A = 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad \text{factoring 3 from the first row}$$
$$= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & -3 \end{bmatrix} \qquad \text{subtracting multiples of the first row from the others}$$
$$= 3 \cdot \det \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & -8 \end{bmatrix} \qquad \text{adding 5 times the second row to the third}$$
$$= 3 \cdot 1 \cdot 1 \cdot (-8) = -24.$$

$$\det B = \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 1 & 2 & 5 \\ 0 & -1 & -1 & -10 \end{bmatrix}$$
$$= \det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & 1 & -15 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 10 \end{bmatrix} \qquad \text{flipped row 3 and 4}$$
$$= -1 \cdot 1 \cdot 1 \cdot 10 = -10.$$

□