Math 2130 - Assignment 9

Due October 29, 2021

(1) Let P_3 the vector space of polynomials of degree ≤ 3 over $\mathbb{R}$ with basis $B = (1, x, x^2, x^3)$.
   (a) Find the matrix $d_{B\leftarrow B}$ for the derivation map $d\colon P_3 \to P_3, p \to p'$.
   (b) Use $d_{B\leftarrow B}$ to compute $[p']_B$ and p' for the polynomial p with $[p]_B = (-3, 2, 0, 1)$.

(2) Let $B = (\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix})$ and $C = (\begin{bmatrix}2\\5\end{bmatrix}, \begin{bmatrix}1\\3\end{bmatrix})$ be bases of $\mathbb{R}^2$, let E be the standard basis of $\mathbb{R}^2$.
   (a) Find the change of coordinates matrix $P_{E\leftarrow B}$ for $f : [u]_B \mapsto [u]_E$.
   (b) Find the change of coordinates matrix $P_{C\leftarrow E}$ for $g : [u]_E \mapsto [u]_C$.
   (c) Find the change of coordinates matrix $P_{C\leftarrow B}$ for $h : [u]_B \mapsto [u]_C$.
   Hint: h is the composition of g and f, $h([u]_B) = g(f([u]_B))$.

(3) Determine the standard matrix for the reflection t of $\mathbb{R}^2$ at the line $3x + y = 0$ as follows:
   (a) Find a basis B of $\mathbb{R}^2$ whose vectors are easy to reflect.
   (b) Give the matrix $t_{B\leftarrow B}$ for the reflection with respect to the coordinate system determined by B.
   (c) Use the change of coordinate matrix to compute the standard matrix $t_{E\leftarrow E}$ with respect to the standard basis $E = (e_1, e_2)$.

(4) (a) Determine the standard matrix A for the rotation r of $\mathbb{R}^3$ around the z-axis through the angle $\pi/3$ counterclockwise.
   Hint: Use the matrix for the rotation around the origin in $\mathbb{R}^2$ for the xy-plane. What happens to e_3 under this rotation?
   (b) Consider the rotation s of $\mathbb{R}^3$ around the line spanned by $\begin{bmatrix}1\\2\\3\end{bmatrix}$ through the angle $\pi/3$ counterclockwise. Find a basis of $\mathbb{R}^3$ for which the matrix $s_{B\leftarrow B}$ is equal to A from (a).
   (c) Give the standard matrix $s_{E\leftarrow E}$ for the standard basis E (You do not need to actually multiply and invert the involved matrices; the product formula is enough).

(5) Compute the determinant of the matrices by cofactor expansion. Pick a row or column that yields the least amount of computation:

$$A = \begin{bmatrix}0 & 1 & -3\\5 & 4 & -4\\0 & -3 & -4\end{bmatrix} \qquad B = \begin{bmatrix}1 & 0 & -3 & 0\\3 & 1 & 5 & 1\\2 & 0 & 0 & 0\\7 & 1 & -2 & 5\end{bmatrix}.$$

(6) **Rule of Sarrus for the determinant of 3×3-matrices.** Let

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\a_{21} & a_{22} & a_{23}\\a_{31} & a_{32} & a_{33}\end{bmatrix}$$

Prove that

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

Hint: Expand det A across the first row.

(7) Consider $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(a) How does switching the rows effect the determinant? Compare det A and det $\begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

(b) How does multiplying one row by a scalar effect the determinant? Compare det A and det $\begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$.

(c) How does adding a multiple of one row to the other row effect the determinant? Compare det A and det $\begin{bmatrix} a & b \\ c+ra & d+rb \end{bmatrix}$.

(8) Compute the determinants by row reduction to echelon form:

$$A = \begin{bmatrix} 3 & 3 & -3 \\ 3 & 4 & -4 \\ 2 & -3 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 3 & 2 & -4 \\ 0 & 1 & 2 & -5 \\ 2 & 7 & 6 & -3 \\ -3 & -10 & -7 & 2 \end{bmatrix}$$